A

SUPPLEMENT

TO

THE ELEMENTS

OF

EUCLID.

SECOND EDITION, CORRECTED AND ENLARGED:

WITH AN APPENDIX OF

MISCELLANEOUS GEOMETRICAL PROPOSITIONS,

DIVIDED INTO SIX BOOKS.

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PREFACE.

The propositions contained in the following compilation are either obvious deductions from those of Euclid, or such as exhibit some remarkable properties of lines, angles, or figures, which are not to be found in Euclid's work; or, lastly, they are the geometrical solutions of many well-known problems in the different branches of Natural Philosophy. But although the propositions, which have here been collected for the use of the academical student, are of these three kinds, it has not been thought advisable to class them according to that threefold division. Designed as a supplement to the Elements of Euclid, those, which constitute the first Part, have been disposed according to Euclid's arrangement. And not only have the propositions contained in the first book been made to depend upon the first book of the Elements, and so on; but the propositions in each separate book will be found, also, to follow the order of the propositions of the corresponding book of Euclid. There is no necessity, therefore, for the student to wait until he has gone through

Euclid's Elements, before he enters upon the perusal of the first Part of this Supplement. It will, perhaps, be more to his advantage to read the original work and this, which is principally intended to supply its deficiencies, together; especially if he has the assistance of a tutor, who will point out to him those theorems and problems which may be considered as best deserving his attention. Some regard has, indeed, been paid to the probability of such a plan being thought worthy of adoption, in the distribution of the matter of this present publication. An endeavour has been made to offer something to the notice of the reader, after almost every one of the most important propositions, in each of the books of Euclid's Elements: so that, supposing him not to advance beyond the first book, or beyond the first four books, of Euclid, a field, more or less contracted, is still open to his research, for the exploring of which he will find himself already sufficiently furnished with previous knowledge. With this view, especially, many of the following propositions, which might undoubtedly have been demonstrated more concisely, if they had been put after Euclid's fifth book, have had a place assigned to them nearer to the beginning. For thus is the learner shewn how extensive an application may be made of some of the simplest elements of Geometry; and thus is a scope afforded to the study of those, who cannot, at first, encounter, without reluctance, the somewhat abstruse reasonings, upon which the ancients, with so much acuteness and solidity of judgment, have founded the doctrine of proportionality.

In order to facilitate the execution of the plan here recommended, an index has been constructed, by means of which the Geometry of the first Part of this Supplement may be incorporated, as it were, with that of Euclid, and the reading of both the treatises may be made to go on together.

In the Second Part, which has been added to this edition, the arrangement has not been made to follow that of Euclid. This part may, therefore, be considered as a separate collection of geometrical propositions, promiscuous in its composition, but yet admitting of a certain degree of classification, which the reader will find adopted in it. Its last two Books relate to subjects, which are almost wholly omitted in the greater number of treatises on Geometry, but which are not deficient in interest.

A general Index to the whole work has, also, been annexed to this second edition, exhibiting the enunciations of all its Propositions, apart from their proofs; in order that the student may use it, not as a mere table of contents only, but as manual problems; not having recourse to the printed demonstrations, until he has exercised his own ingenuity in discovering solutions.

In the demonstrations of the propositions recourse has been had to symbols. But these symbols are merely the representatives of certain words and phrases, which may be substituted for them at pleasure, so as to render the language employed strictly conformable to that of ancient Geometry. The consequent diminution of the bulk of the whole book is the least advantage which results from this use of symbols. For the demonstrations themselves are sooner read and more easily comprehended by means of these useful abbreviations; which will, in a short time, become familiar to the reader, if he is not beforehand perfectly well acquainted with them.

It appeared to be unnecessary to print the formal and logical conclusion which belongs to every geometrical demonstration, and which consists in repeating the enunciation of the proposition which was to be proved, and in asserting that it has been proved. This last step, is, therefore, left for the reader in all cases mentally to supply. And if some omissions of a weightier kind, and some errors, be discoverable in the following pages, it is hoped that they will be found neither too great, nor too many to be forgiven, if the general plan of the work meet with the approbation of those who are competent to decide upon it.

Vicarage-House,
Enfield.

AN INDEX,

Shewing the Order in which the Propositions of the first Part of the following Supplement may be read along with the Propositions contained in Euclid's Elements.

BOOK. I.

After	E. 9.	1.	may be read	S. 1.	1.
———	E. 10.	1.	———	S. 2.	—
———	E. 11.	1.	———	S. 3. S. 4. S. 5.	— — —
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———	E. 43.	1.	———	S. 70.	—
———	E. 45.	1.	———	S. 71. S. 72.	— —
———	E. 47.	1.	———	S. 73. S. 74. S. 75. S. 76. S. 77.	— — — — —

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——	E. 12.	6.	——	S. 40.	—
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——— E. 20. 6. ———	S. 63. — S. 64. — S. 65. — S. 66. —
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AN EXPLANATION

OF THE SYMBOLS EMPLOYED IN THIS TREATISE, AS ABBREVIATIONS.

$=$	denotes	*is equal to* or *equal to.*
$>$		*is greater than.*
$<$		*is less than.*
$+$		*together with.*
$-$		*diminished by.*
$\angle$		*angle.*
$\angle$s		*angles.*
AB, or $\overline{AB}$		*a straight line, of which the points denoted by* A *and* B *are the extremities.*
$\overset{\frown}{AB}$		*a circular arch, of which the points denoted by* A *and* B *are the extremities.*
$\overline{AB}^2$		*a square, having* $\overline{AB}$ *for one of its sides.*
$\overline{AB} \times \overline{CD}$		*a rectangle, of which* $\overline{AB}$ *and* $\overline{CD}$ *are adjacent sides.*
$2\,\overline{AB}$, &c.		*the double, &c. of* AB.
$\triangle$		*a triangle.*
$\triangle$s		*triangles.*
▭		*a parallelogram.*
▭s		*parallelograms.*
A : B		*the ratio of* A *to* B.
A : B :: C : D		*the ratio of* A *to* B *is equivalent to the ratio of* C *to* D.
$\therefore$		*therefore.*

A

SUPPLEMENT

TO THE

ELEMENTS OF EUCLID.

PART I.

BOOK I.

Prop. I.

1. **Problem.** *A given plane rectilineal angle being divided into any number of equal angles, to divide the half of it into the same number of angles, all equal to one another.*

Bisect (E.* IX. 1.) the given angle: And, first, if it be divided into an *odd* number of equal parts, it is evident that the middle part is thereby bisected. Bisect, therefore, each of the remaining equal parts, on either side of that middle part, and the half of the given angle will, manifestly, be divided into as many equal parts as the given angle itself.

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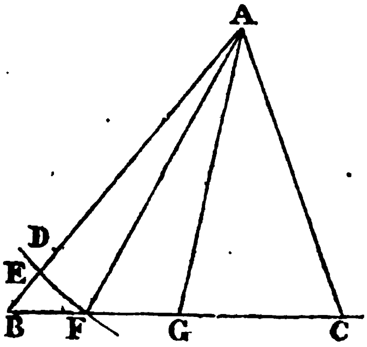
* In this and the following references, the letter E is used to indicate *Euclid's Elements;* the letter S, in like manner, refers to this *Supplement;* the former of the subsequent numbers points out the *Proposition,* and the latter the *Book,* intended to be quoted.

Again, if the given angle be divided into an *even* number of equal parts, it is plain that the straight line which bisects it, will have the half of that number of equal parts, on each side of it. Bisect, therefore, each of the equal parts, on either side of that line; and the half of the given angle will thereby be divided, as before, into as many equal parts as the given angle itself.

PROP. II.

2. PROBLEM. *From the vertex of a given scalene triangle, to draw, to the base, a straight line which shall exceed the less of the two sides, as much as it is itself exceeded by the greater.*

Let *ABC* be the given scalene triangle, and let *AB* be greater than *AC*: It is required to draw,



from the vertex *A*, to the base *BC*, a straight line which shall exceed *AC*, as much as it is exceeded by *AB*.

From *AB* cut off (E. iii. 1.) *AD* = *AC*; bisect (E. x. 1.) *DB* in *E*; from the centre *A*, at the distance AE, describe (E. III. *Post.*) the circle EF cutting BC in F; and join (E. I. *Post.*) A, F: Then is AF the straight line which was to be drawn.

For, (E. XV. Def. 1.) $AF = AE$; and (*constr.*) $AD = AC$; $\therefore AF - AC = AE - AD = DE$.

Also, $AB - AE = BE$; *i.e.* $AB - AF = BE$: and (*constr.*) $BE = DE$;

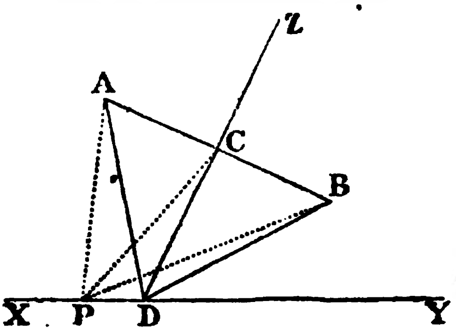
$$\therefore AF - AC = AB - AF.$$

PROP. III.

3. PROBLEM. *In a straight line given in position, but indefinite in length, to find a point, which shall be equidistant from each of two given points, either on contrary sides, or both on the same side of the given line, and in the same plane with it; but not situated in a perpendicular to it.*

Let XY be an indefinite straight line, and A, B, two points without it; not situated in a perpendicular to XY: It is required to find a point in XY that shall be equidistant from A and B.

First, let A, B be both on the same side of XY:



Join A, B; bisect (E. X. 1.) AB in C; from C draw (E. XI. 1.) CD perpendicular to AB, meeting XY in D. The point D is equidistant from A, B.

For, join *A, D* and *B, D*. Then, since (*constr.*) *AC*=*BC*, and *CD* is common to the two △s *ACD, BCD*, and that (*constr.* and E. x. Def. 1.) ∠ *ACD* = ∠ *BCD*; ∴ (E. IV. 1.) *AD* = *BD*; *i. e. D* is equidistant from *A* and *B*.

But, if the two given points, *A* and *B*, are on contrary sides of *XY*, let them be joined, as before, and let the straight line which joins them be bisected.

Then, if the point of bisection be in *XY*, that, which was required, has been done, But, if that point be not in *XY*, draw from it, as before, a perpendicular to *AB*, and it may be shown, as in the first case, that the point, in which the perpendicular meets *XY*, is that which was required to be found.

4. COR. 1. By the help of this problem, it is manifest that a circle may be described, which shall have its centre in a given straight line, and which shall pass through two given points without that line.

5. COR. 2. It is evident from the demonstration, that any point in an indefinite straight line *DZ*, which bisects the given finite straight line *AB*, at right angles, is equidistant from the extremities *A* and *B*, of that given finite line: And, any point which is not in that indefinite line *DZ*, is not equidistant from the two extremities *A* and *B* of the given finite line.

For, let *P* be any point, not in *DZ*, which bisects *AB* at right angles in *C*; and, if it be possible, let *P* be equidistant from *A* and *B*: Join *P, A* and *P, C* and *P, B*; and since (*hyp.*) *AC*=*CB*, and *CP* is common to the two △s *ACP, BCP*, and that (*hyp.*) *PA* = *PB*; ∴ (E. VIII. 1.) the ∠ *ACP*= ∠ *BCP*, and ∴ (E. x. Def. 1.) the ∠ *ACP* is a right angle; but (*hyp.*) the ∠ *ACD* is a right angle; therefore the ∠ *ACP* is equal to the ∠ *ACD*, the less to the greater, which is impossible; therefore the point *P* is not equidistant from *A* and *B*.

6. Cor. 3. Hence, an indefinite number of circles may be described all of them passing through two given points: And if any number of circles pass, all of them, through the same two given points, their centres are all in the straight line that bisects at right angles the straight line joining the two given points.

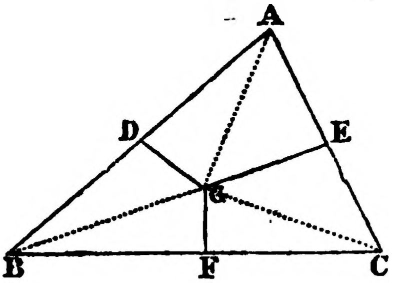
7. Cor. 4. Hence, also, a circle may be described which shall pass through two given points, and which shall have its semi-diameter equal to any given finite straight line, that exceeds the half of the straight line joining the two given points.

For, let *A*, *B* be the two given points; and join *A*, *B*; and let *CD* be drawn bisecting *AB* at right angles; from *A*, as a centre, at a distance equal to the given finite straight line, describe a circle, and let it cut *CD* in *D*; therefore, (Cor. 2.) *D* is equidistant from *A* and *B*; and therefore a circle described from *D*, as a centre, at the distance *DA*, which (*constr.* E. xv. Def. 1.) is equal to the given semi-diameter, will pass through *B*.

Prop. IV.

8. Theorem. *If the three sides of a given triangle be bisected, the perpendiculars drawn to the sides, from the three several bisections, shall all meet in the same point: And that point is equidistant from the three angular points of the given triangle.*

Let *ABC* be a triangle, of which the three sides *AB*, *AC*, and *CB* are bisected in the points *D*, *E* and *F*, respectively: The perpendiculars drawn to the several sides from *D*, *E*, *F*, shall all meet in a point that is equidistant from *A*, *B* and *C*.



For, draw (E. xi. 1.) *DG* perpendicular to *AB*, and *EG* perpendicular to *AC*, and let them meet in *G*: Join *G*, *F*. Then, (*constr.* and Sect. iii. 1. Cor. 2.) ∴ $BG = AG$, and $AG = CG$; ∴ $CG = BG$.

Again, since (*hyp.*) $BF = CF$ (*constr.*) and *FG* is common to the two △s *BFG*, *CFG*, and that $BG = CG$; therefore the $\angle BFG = \angle CFG$ (E. viii. 1); *i.e.* (E. x. Def. 1.) *GF* is perpendicular to *BC*: And there cannot (E. x. Def. 1.) be drawn from *F* more than one straight line perpendicular to *BC*. It is plain, therefore, that the perpendiculars drawn to the sides, from *D*, *E* and *F*, all meet in the same point *G*: And, since it has been shown that $AG = BG = CG$, the point *G* is equidistant from *A*, *B* and *C*.

Prop. V.

9. Problem. *To find a point, in a given plane, which shall be equidistant from three given points in the plane, that are not all in the same straight line.*

Let A, B, C, (See Fig. p. 6.) be three points, not all of them in the same straight line: It is required to find a point, that shall be equidistant from A, B and C.

Join A, B, and B, C, and C, A; bisect (E. x. 1.) AB in D, and AC in E; draw (E. xi. 1.) from D and E, DG perpendicular to AB, and EG perpendicular to AC, and let them meet in G.

Then, (Sect. iii. 1. Cor. 2.) the point G is equidistant from A, B, and C.

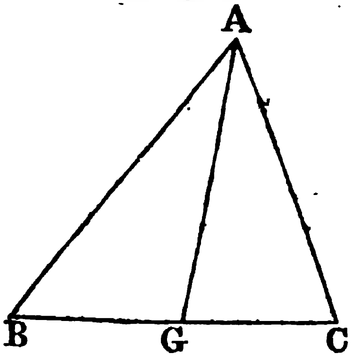
10. Cor. By the help of this problem a circle may be described about a given triangle; or so as that its circumference shall pass through any three given points that are not in the same straight line.

Prop. VI.

11. Theorem. *There cannot be drawn more than two equal straight lines, to another straight line, from a given point without it.*

Let A be a point, without the straight line BC: There cannot be drawn more than two equal straight lines, from A to BC.

For, if it be possible, let $\overline{AB}=\overline{AG}=\overline{AC}$; therefore (E. v. 1.) $\angle ACB = \angle AGC$: Also $\angle ACB =$



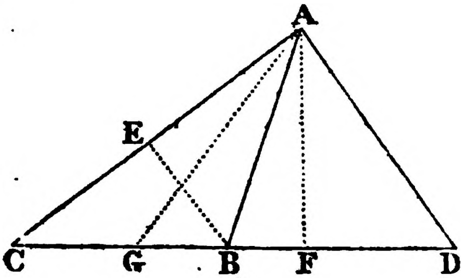
∠ *ABC*; ∴ ∠ *AGC* = ∠ *ABC*; *i. e.* the exterior is equal to the interior opposite angle, when the side *BG*, of the △ *AGB*, is produced: which (E. XVI. 1.) is absurd.

12. COR. A circle cannot cut a straight line in more points than two.

PROP. VII.

13. THEOREM. *The perpendicular let fall from the obtuse angle of an obtuse-angled triangle, or from any angle of an acute-angled triangle, upon the opposite side, falls within that side: But the perpendicular drawn to either of the sides containing the obtuse angle of an obtuse-angled triangle, from the angle opposite, falls without that side.*

Let *ABC* be an obtuse-angled triangle, obtuse-angled at *B*, and let *ABD* be an acute-angled triangle: The perpendicular drawn from *B* to *AC* falls within *AC*; the perpendicular drawn from any other ∠ *A*, of the △ *ABC*, to the opposite side *BC*, falls without *BC*; and the perpendicular from any ∠ *A*, of the △ *ABD*, to the opposite side *BD*, falls within *BD*.



For, first, if it be possible, let *AG*, drawn (E. XII. 1.) from *A* perpendicular to *BD*, meet *DB*, produced, in *G*: Then, since (*hyp.*) the ∠ *ABD* is acute, the ∠ *ABD* is (E. XIII. 1.) obtuse; and (*constr.*) the ∠ *AGB* is a right angle: Wherefore the two ∠s *ABG*, *AGB* of the △ *ABG* are not less than two right angles; which (E. XVII. 1.) is absurd. Therefore, the perpendicular drawn from *A* on *BD* cannot fall without *BD*. And, in the same manner, it may be shewn, that the perpendicular drawn from *B* on the opposite side *AC*, of the obtuse-angled △ *ABC*, cannot fall without *AC*, and also that the perpendicular drawn from *A*, on the opposite side *BC*, of that triangle, cannot fall within *BC*.

PROP. VIII.

14. THEOREM. *If a straight line, meeting two other straight lines, makes the two interior angles on the same side of it not less than two right angles, these lines shall never meet on that side, if produced ever so far.*

For, if it be possible, let two straight lines meet; which make, with another straight line, the two interior angles, on the same side, not less than two right angles: Then it is plain, that the three straight lines will thus include a triangle, two angles of which are not less than two right angles; which (E. XVII. 1.) is absurd. Wherefore, the two straight lines cannot meet, on that side of the straight line, on which they make the two interior angles not less than two right angles.

15. COR. Two straight lines, which are both perpendicular to the same straight line, are parallel to each other.

Prop. IX.

16. Theorem. *The three sides of a triangle taken together, exceed the double of any one side, and are less than the double of any two sides.*

For, since (E. xx. 1.) any two sides of a triangle are greater than the third, if the third side be added both to those two and to itself; it is evident that the three sides are, together, greater than the double of the third.

Again, since (E. xx. 1.) any side of a triangle is less than the other two, if the other two be added both to that side, and to themselves, it is evident, that the three sides are, together, less than the double of the other two.

Prop. X.

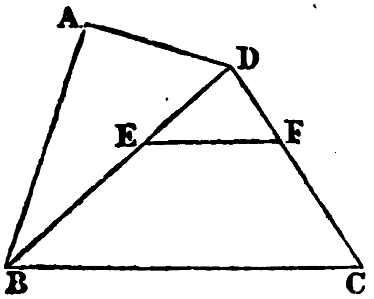
17. Theorem. *Any side of a triangle is greater than the difference between the other two sides.*

If, the triangle be equilateral, or isosceles, the proposition is manifestly true. But let it be a scalene triangle: Then, since (E. xx. 1.) any two sides of the triangle are greater than the third, if either of those two be taken from that third side, it is plain that the remaining side is greater than the difference of the other two.

Prop. XI.

18. Theorem. *Any one side of a rectilineal figure is less than the aggregate of the remaining sides.*

Let *ABCD* be a rectilineal figure: Any one



side, as *BC,* is less than the aggregate of the remaining sides.

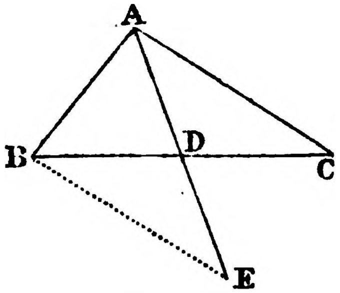
For, first, let the figure be quadrilateral; and join *B, D*: Then (E. xx. 1.) $BD+DC>BC$; and, $BA+AD>BD$; $\therefore$ $BA+AD+DC>BD+DC$; much more, then, is $BA+AD+DC>BC$.

And the proposition may, in the same manner, be proved to be true, when the figure has more than four sides.

PROP. XII.

19. THEOREM. *The two sides of a triangle are together, greater than the double of the straight line which joins the vertex and the bisection of the base.*

Let *ABC* be a triangle, and let *AD* be the

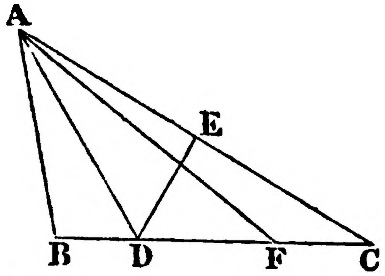


straight line joining the vertex *A*, and the bisection, *D*, of the base *BC* : $AB+AC>2AD$. Produce *AD* to *E*, and cut off (E. III. 1.) $DE=AD$; also, join *B*, *E*.

Then since (*hyp.*) $BD=BC$, and (*constr.*) $AD=DE$, the two sides *BD*, *DE*, of the △*BDE*, are equal to the two sides *AD*, *DC* of the △*ADC*; and (E. XV. 1.) the ∠*BDE* = ∠*ADC*; ∴ (E. IV. 1.) $BE=AC$. But (E. XX. 1.) $AB+BE>AE$; but *AC* has been proved to be equal to *BE*, and *AE* is (*constr.*) the double of *AD*; ∴ $AB+AC>2AD$.

PROP. XIII.

20. THEOREM. *The two sides of a triangle are, together, greater than the double of the straight line drawn from the vertex to the base, bisecting the vertical angle.*



Let *ABC* be a triangle, and let *AD* be drawn from the vertex *A*, to the base *BC*, bisecting the vertical ∠*BAC*: Then, $AB+AC>2AD$.

If the triangle be isosceles, the straight line which bisects the vertical angle is (E. IV. 1.) perpendicular to the base; and since (E. XVII. 1. and E. XIX. 1.) each of the equal sides is greater than the perpendicular, the proposition, is, in this case, manifestly true.

But, let ABC be a scalene triangle, and let the side AB be less than AC: Then, of the segments into which AD, bisecting the $\angle BAC$, divides the base BC, BD, which is adjacent to the less side AB, is the less.

For, from AC, the greater, cut off (E. III. 1.) $AE=AB$, the less, and join D, E; and because BA, AD are equal to EA, AD, and (*hyp.*) the $\angle BAD = \angle EAD$; $\therefore$ (E. IV. 1.) $BD=DE$, and $\angle BDA=\angle EDA$; but (E. XVI. 1.) $\angle DEC>\angle ADE$; $\therefore \angle DEC>\angle ADB$; and (E. XVI. 1.) $\angle ADB>\angle ACD$; much more then is $\angle DEC>\angle ECD$; $\therefore$ (E. XIX. 1.) $DC>DE$; but it has been shewn that $DE=DB$; $\therefore$ $DC>DB$. From DC, the greater cut off (E. III. 1.) $DF=DB$; and join A, F: Then (E. XVI. 1.) the $\angle AFC>\angle ABC$; and because (*hyp.*) $AC>AB$, $\therefore$ (E. XVIII. 1.) $\angle ABC>\angle ACB$; much more then is $\angle AFC>\angle ACF$; $\therefore$ (E. XIX. 1.) $AC>AF$: But (S. xii. 1. and *constr.*) $AB+AF>2AD$; much more then is $AB+AC>2AD$.

21. Cor. From the demonstration it is manifest, that of the segments into which the straight line bisecting the vertical angle of a scalene triangle, divides the base, that which is adjacent to the less side, is the less.

Prop. XIV.

22. Theorem. *If a trapezium and a triangle stand upon the same base, and on the same side of it, and the one figure fall within the other, that which has the greater surface shall have the greater perimeter.*

Let the trapezium *EBCF* (See Fig. p. 11.) fall within the △*DBC*; let, also, the △ *DBC* fall within the trapezium *ABCD*; and let all the figures stand on the same base *BC*: The perimeter of the △*DBC* is greater than the perimeter of *EBCF*, and less than the perimeter of *ABCD*.

First, let *E* and *F* be in the sides *DB* and *DC* of the △*DBC*, and let the vertex *D* of the △*DBC* coincide with the ∠ *A* or the ∠ *D* of the trapezium *ABCD*.

Then, since (E. xx. 1.) $DE+DF>EF$, add to both, *EB*, *BC*, and *CF*; ∴ $DE+EB+DF+FC+BC>EF+FC+CB+BE$; *i. e.* the perimeter of the △*DBC* >the perimeter of *EBCF*.

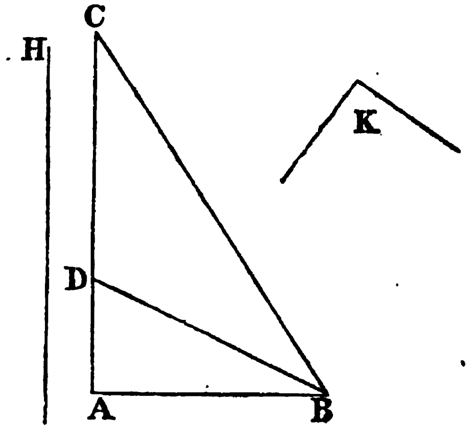
Again, since (E. xx. 1.) $BA+AD>BD$, add to both *DC* and *CB*; ∴ $BA+AD+DC+CB>BD+DC+CB$; *i. e.* the perimeter of the trapezium *ABCD*> the perimeter of the △*DBC*.

And, if *E* or *F* fall within the △*DBC*, and the vertex of the triangle do not coincide with either of the ∠s *A* or *D*, of the trapezium, it may, in the same manner, be proved, that the proposition is true, a fortiori.

PROP. XV.

23. PROBLEM. *One of the angles at the base of a triangle, the base itself, and the aggregate of the two remaining sides, being given, to construct the triangle.*

Let *K* be the given angle, *AB* the given base of the triangle, and *H* the aggregate of the two



remaining sides: It is required to construct the triangle.

At the point *A*, in *AB*, make (E. XXIII. 1.) the $\angle BAC = \angle K$, and make (E. III. 1.) $AC = H$; join *C*, *B*; and at the point *B*, in *CB*, make (E. XXIII. 1.) the $\angle CBD = \angle ACB$: Then is *DAB* the triangle which was to be constructed.

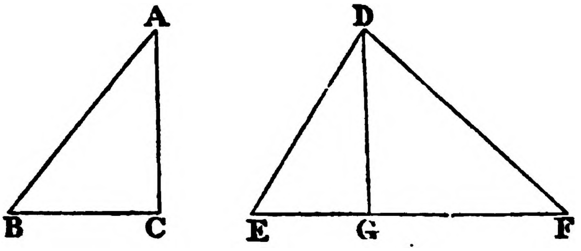
For, since (*constr.*) $\angle DCB = \angle DBC$, $\therefore$ (E. VI. 1.) $BD = DC$; add to both DA; $\therefore$ $BD + DA = CD + DA$; *i. e.* $BD + DA = CA$; and (*constr.*) $CA = H$; $\therefore$ $BD + DA = H$; and the $\angle A$ was made equal to the given $\angle K$: It is manifest, therefore, that *DAB* is the triangle which was to be constructed.

PROP. XVI.

24. THEOREM. *If two right-angled triangles have the three angles of the one equal to the three angles of the other, each to each, and if a side of the one be equal to the perpendicular let fall from the right*

angle upon the hypotenuse of the other, then shall a side of this latter triangle be equal to the hypotenuse of the former.

Let *ACB* and *EDF* be two right-angled triangles, right-angled at *C* and *D*, having, also the ∠*DEF*= ∠*ABC*, the ∠*EFD* = ∠*CAB*, and the side *AC*, of the △*ABC*, equal to the perpendicular *DG*, drawn from *D* to the hypotenuse *EF* of the △*DEF*: The side *DE*, of the △*DEF*, is equal to the hypotenuse *AB*, of the △*ABC*.

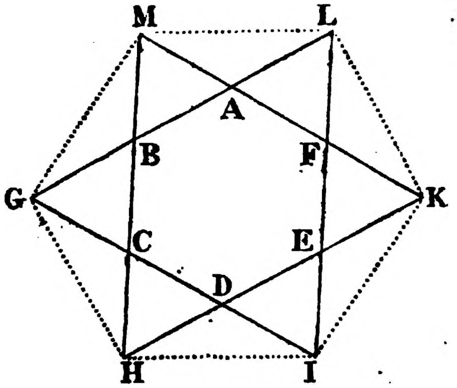


For, since *AC*=*DG*, and the two ∠s *ACB*, *ABC*, of the △*ABC*, are equal to the two angles, *DGE*, *DEG*, of the △*DEG*, each to each, ∴ (E. XXVI. 1.) *DE*=*AB*.

Prop. XVII.

25. Theorem. *If the sides of any given equilateral and equiangular figure of more than four sides, be produced so as to meet, the straight lines, joining their several intersections, shall contain an equilateral and equiangular figure, of the same number of sides as the given figure.*

Let *ABCDEF* be any equilateral and equiangular figure, of more than four sides; let the sides, produced, meet in the points $G, H, I, K,$



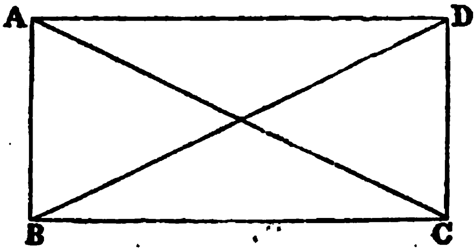
L, M; and let those points of intersection be joined: Then is $GHIKLM$ an equilateral and equiangular figure, of the same number of sides as $ABCDEF$.

For, since (*hyp.*) the $\angle$s A, B, C, D, E, F are all equal, the $\triangle$s $MAB, GBC, HCD, IDE, KEF, LFA$ are all (E. xiii. 1. and E. vi. 1.) isosceles, and any two of them have their angles equal, each to each; ∴ since (*hyp.*) $BA = AF$, and that the $\angle$s MAB, MBA are equal to the $\angle$s LFA, LAF, each to each, the side MA of the $\triangle MAB =$ the side LA (E. xxvi. 1,) of the $\triangle LAF$; and in the same manner it may be shewn that $MB = GB$, $GC = CH$, $HD = DI$, $IE = EK$, and $KF = FL$: But, because the angles of the figure $ABCDEF$ are (*hyp.*) equal, therefore (E. xv. 1.) the $\angle$s $LAM, MBG, GCH, HDI, IEK, KFL$, are all equal to one another; therefore (E. iv. 1.) the sides LM, MG, GH, HI, IK and KL are all equal, as are also the angles of the $\triangle$s LAM, MBG, GCH, HDI, IEK, and KFL, each to each: And the $\angle$s AMB, BGC, CHD, DIE, EKF, and FLA have been shewn to be equal to one another: Wherefore the figure *GHIKLM* is equilateral and equiangular; and it is manifest that it has the same number of sides as the figure *ABCDEF*.

PROP. XVIII.

26. THEOREM. *If two opposite sides of a quadrilateral figure be equal to one another, and the two remaining sides be also equal to one another, the figure is a parallelogram.*

Let any two opposite sides, as *AB*, *DC*, of the quadrilateral figure *ABCD*, be equal to one another, and let the two remaining sides, *AD*, *BC*, be, also, equal to one another: The figure *ABCD* is a parallelogram.



For, join *D*, *B*: Then since the two sides *AD*, *DB*, of the △ *ADB*, are equal to the two sides *CB*, *BD*, of the △ *CBD*, and that the base *AB* is equal (*hyp.*) to the base *DC*; therefore (E. VIII. 1.) the ∠ *ADB* = ∠ *DBC*; and (E. IV. 1.) the ∠ *ABD* = ∠ *BDC*; therefore (E. XXVII. 1.) *AD* is parallel to *BC*, and *AB* is parallel to *DC*; *i.e.* the figure *ABCD* is a parallelogram.

27. COR. 1. Hence may be deduced a practical method of drawing a straight line, through a given point, parallel to a given straight line.

For, let it be required to draw through the given point *B*, a straight line parallel to *AD*: From any point *A* in *AD*, as a centre, and at any distance, describe a circle cutting *AD* in *D*; and from *B* as a centre, at the same distance, describe another circle; lastly, from *D* as a centre, at a distance equal to that of *A*, *B*, describe another circle, cutting the circle last described in *C*; join *B*, *C*. *BC* is parallel to *AD*.

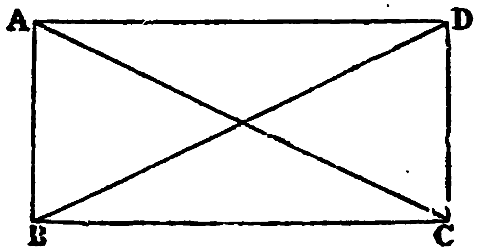
For, if *A*, *B*, and *D*, *C* be joined, it is manifest from the construction, that $AD = BC$, and $AB = DC$: therefore (Supp. XVI. 1.) *BC* is parallel to *AD*.

28. Cor. 2. A rhombus is a parallelogram.

Prop. XIX.

29. Theorem. *Every parallelogram which has one angle a right angle, has all its angles right angles.*

Let one angle, as *A*, of the ▭ *ABCD* be a right angle: The ∠s *B*, *C*, and *D* are also right angles.

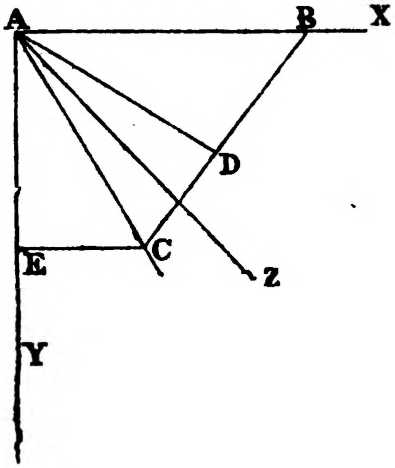


For, since *AD* is parallel to *BC*, and *AB* meets them, the two interior ∠s *A*, *B* are, (E. XXIX. 1.) together, equal to two right angles; but (*hyp.*) the ∠ *A* is a right angle; therefore the ∠ *B* is also a right angle: And, in the same manner, may the remaining angles, *C* and *D*, be shewn to be right angles.

Prop. XX.

30. Problem. *To trisect a right angle; i. e. to divide it into three equal parts.*

Let the ∠ *XAY* be a right angle: It is required to trisect it; *i. e.* to divide it into three equal parts.



In *AX* take any point *B*; upon *AB* describe (E. i. 1.) the equilateral △ *ACB*; and from *A* draw (E. xii. 1.) *AD* perpendicular to *BC*: The ∠ *XAY* is trisected by the two straight lines *AC* and *AD*.

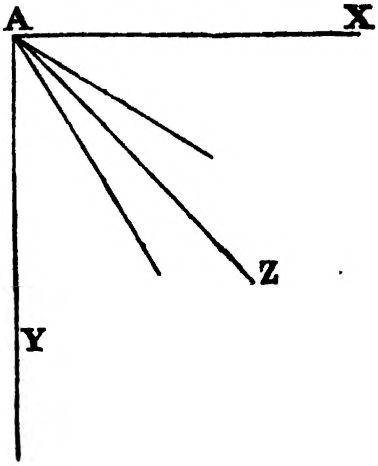
For, from *C* draw (E. xii. 1.) *CE* perpendicular to *AY*; then, since the ∠s *BAE*, *AEC*, are right angles; therefore (E. xxviii. 1.) *AB* is parallel to *EC*; therefore (E. xxix. 1.) ∠ *ECA* = ∠ *CAB* = ∠ *ACB*; because (*constr.*) the △ *ACB* is equilateral, and (E. v. 1. *Cor.*) equiangular: Since, therefore, the ∠ *ACE* = ∠ *ACD*, and that the ∠s *D* and *E* are right angles, and *AC* is common to the two △s *ADC*, *AEC*; therefore (E. xxvi. 1.) the ∠ *EAC* = ∠ *DAC*: Again, since (*constr.* and E. v. 1. *Cor.*) the ∠ *ACB*

= ∠ *ABC*, and (*constr.*) the angles at *D* are right angles, and that *AC*=*AB*; therefore (E. XXVI. 1.) the ∠ *DAC*= ∠ *DAB*: But it was shewn that the ∠ *EAC*= ∠ *DAC*; ∴ ∠ *EAC*= ∠ *DAC*= ∠ *DAB*; *i. e.* the right ∠ *XAY* is trisected by *AC* and *AD*.

Prop. XXI.

31. Problem. *Hence, to trisect a given rectilineal angle, which is the half, or the quarter, or the eighth part, and so on, of a right angle.*

First, let the given ∠ *YAZ*, be the half of a right angle, and let it be required to trisect it.



Draw (E. XI. 1.) from *A*, *AX* perpendicular to *AY*; trisect (Supp. XVIII. 1.) the right ∠ *XAY*; then (Supp. I. 1.) trisect the ∠ *YAZ*, which is the half of the ∠ *YAX*.

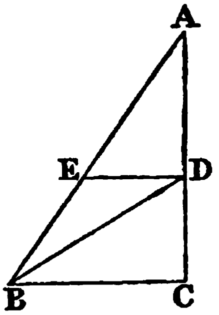
But, if the given angle be the quarter of a right angle, its double may be trisected by the former case; and therefore the given angle itself may be trisected by (Supp. I. 1.).

And, by following the same method, it is evident that an angle may be trisected, which is the eighth part, or the sixteenth part, and so on, of a right angle.

PROP. XXII.

32. PROBLEM. *In the hypotenuse of a right-angled triangle, to find a point, the perpendicular distance of which from one of the sides, shall be equal to the segment of the hypotenuse between the point and the other side.*

Let *ABC* be a right-angled triangle, right-angled



at *C*: It is required to find a point in the hypotenuse *AB*, the perpendicular distance of which from one of the sides, as *AC*, shall be equal to the segment of the hypotenuse between that point, and *BC*.

Bisect (E. IX. 1.) the ∠ *ABC*, by *BD*, and let *BD* meet *AC* in *D*; through *D*, draw *DE* (E. XXXI. 1.) parallel to *CB*: *E* is the point which was to be found.

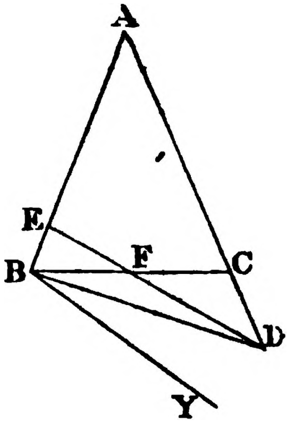
For, since *DE* is parallel to *CB*, the ∠ *CBD* = ∠ *BDE* (E. XXIX. 1.); but (*constr.*) the ∠ *CBD* = ∠ *DBE*; ∴ ∠ *DBE* = ∠ *BDE*; ∴ (E. VI. 1.) *ED* = *EB*; and since (*hyp.*) the ∠ *C* is a right angle, and that DE is parallel to CB, the $\angle CDE$ (E. XXIX. 1.) is a right angle; *i. e.* ED is perpendicular to AC.

PROP. XXIII.

33. PROBLEM. *In the base of a given acute-angled triangle, to find a point, through which if a straight line be drawn perpendicular to one of the sides, the segment of the base, between that side and the point, shall be equal to the segment of the perpendicular, between the point and the other side produced.*

Let ABC be the given acute-angled triangle: It is required, to find, in the base BC, a point through which if a perpendicular be drawn to AB, the segment of the base, between that point and the point B, shall be equal to the segment of the perpendicular between that same point and AC produced.

Draw (E. XI. 1.) from B, BY perpendicular to



AB; bisect (E. IX. 1.) the $\angle CBY$ by BD, meeting AC produced in D; through D, draw (E. XXXI. 1.)

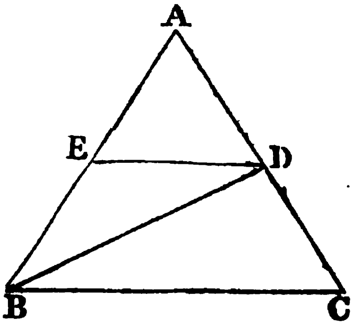
DE parallel to *BY*, and let *DE* cut *BC* in *F*: *F* is the point which was to be found.

For, since (*constr.*) the ∠*ABY* is a right angle, and that *DE* is parallel to *BY*, the ∠*E* (E. XXIX. 1.) is, also, a right angle; and the ∠*YBD*= ∠*BDF*; but (*constr.*) the ∠*YBD*= ∠*DBF*; therefore the ∠*DBF*= ∠*BDF*; therefore (E. VI. 1.) *FB*=*FD*.

PROP. XXIV.

34. PROBLEM. *From a given isosceles triangle to cut off a trapezium, which shall have the same base as the triangle, and shall have its three remaining sides equal to each other.*

Let *ABC* be the given isosceles triangle: It is required to cut off from it a trapezium, which, having *BC* for its base, shall have its three remaining sides equal to one another.



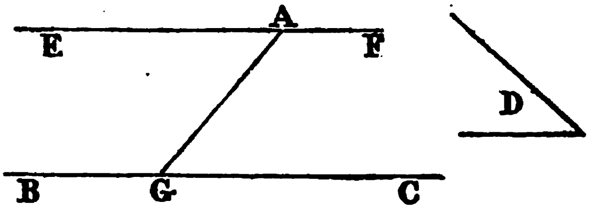
Bisect (E. XIX. 1.) the ∠*ABC* by *BD*, meeting *AC* in *D*; and through *D* draw (E. XXXI.) *DE* parallel to *CB*: Then shall *BE*, *ED*, and *DC*, the three sides of the trapezium *BEDC*, be equal to one another.

For, since *DE* is parallel to *BC*, the ∠*AED*= ∠*ABC* (E. XXIX. 1.), and ∠*ADE* = ∠*ACB*; but (*hyp.* and E. v. 1.) $\angle ABC = \angle ACB$; therefore, $\angle AED = \angle ADE$; therefore (E. vi. 1.) $AE = AD$; but (*hyp.*) $AB = AC$; from these equals take the equals AE, and AD, there remains $EB = DC$: Again, because DE is parallel to BC, the $\angle CBD = \angle BDE$ (E. xxix. 1.); but (*constr.*) $\angle CBD = \angle DBE$; therefore the $\angle DBE = \angle BDE$; therefore (E. vi. 1.) $EB = ED$; and EB has been proved to be equal to DC; $\therefore$ EB, ED and DC are equal to one another.

PROP. XXV.

35. PROBLEM. *To draw to a given straight line, from a given point without it, another straight line which shall make with it an angle equal to a given rectilineal angle.*

Let BC be a straight line, A a point without it, and D a rectilineal angle: It is required to draw from A, a straight line which shall make with BC an angle equal to the $\angle D$.



Through A draw (E. xxxi. 1.) EAF parallel to BC; at the point A in EAF, make (E. xxiii. 1.) the $\angle EAG = \angle D$: AG is the line which was to be drawn.

For, since (*constr.*) EF is parallel to BC, the $\angle EAG = \angle AGC$ (E. xxix. 1.); but (*constr.*) the $\angle EAG = \angle D$; $\therefore$ $\angle AGC = \angle D$.

PROP. XXVI.

36. THEOREM. *If all the angles but one of any rectilineal figure, be together, equal to all the angles but one, of another rectilineal figure having the same number of sides, the remaining angle of the one figure, shall be equal to the remaining angle of the other: And, conversely, if an angle in the one figure be equal to an angle in the other, the remaining angles of the one shall be equal, together, to the remaining angles of the other.*

For, since the two figures have the same number of sides, all the interior angles of the one are, together, equal. (E. XXXII. Cor. 1.) to all the interior angles of the other: If, therefore, from these equals be taken first, the aggregates which, by the hypothesis, are equal; and secondly, the single angles, which are supposed to be equal, it is manifest that the remaining angle, or angles, of the one figure, must be equal to the remaining angle, or angles of the other.

PROP. XXVII.

37. THEOREM. *The angle at the base of an isosceles triangle is equal to, or is less, or greater, than the half of the vertical angle, accordingly as the triangle is a right-angled, an obtuse-angled, or an acute-angled triangle.*

For, (E. V. 1, and E. XXXII. 1.) the double of the angle at the base together with the vertical angle, is equal to two right angles; if therefore the vertical angle be a right angle, and if it be taken from both, there remains the double of the angle at the base equal to a right angle; therefore the angle at the base is equal to half a right angle; that is, is equal to half the vertical angle.

But, if the vertical angle be obtuse, when it is taken away from the same equals as before, there will remain the double of the angle at the base equal to a less angle than a right angle; therefore the angle at the base is, in this case, less than the half of a right angle; much more, therefore, is it less than half the obtuse vertical angle.

And, in like manner it may be shewn, that, when the vertical angle of the isosceles triangle is acute, the angle at the base is greater than the half of a right angle; much more, therefore, is it greater than half the acute vertical angle.

Prop. XXVIII.

38. Theorem. *If either of the equal sides of an isosceles triangle be produced, towards the vertex, the straight line, which bisects the exterior angle, shall be parallel to the base.*

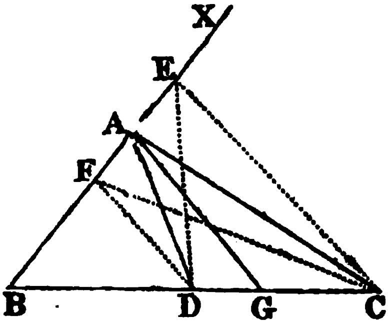
For, (E. v. 1. and E. xxxii. 1.) the exterior angle at the vertex of an isosceles triangle is the double of either of the angles at the base; therefore the half of that interior angle is equal to either of the angles at the base; therefore the straight line bisecting the vertical angle is (E. xxviii. 1.) parallel to the base.

Prop. XXIX.

39. Theorem. *The distance of the vertex of a triangle from the bisection of its base, is equal to, greater than, or less than the half of the base, ac-*

cordingly as the vertical angle is a right, an acute, or an obtuse angle.

First, let *ABC* be a right-angled triangle, right-angled at *A*, and let *AD* join *A* and the bisection, *D*, of the base: *AD*=*DB*, or *DC*.



For, if not, *AD* is either greater or less than *BD*: Produce *BA* to *X*; and first, if it be possible, let *AD*>*DB*; therefore, also, *AD*>*DC*; therefore (E. XVIII. 1.) ∠*B*>∠*BAD*, and ∠*C*>∠*CAD*; ∴ ∠*B*+∠*C*>∠*BAD*+∠*CAD*; *i.e.* ∠*B*+∠*C*>∠*BAC*; but (*hyp.* and E. Def. 10. 1.) ∠*BAC* = *CAX*; ∴ ∠*B*+∠*C*>∠*CAX*; which (E. XXXII. 1.) is absurd.

And, in like manner, if *DA* be supposed to be less than *BD*, it may be shewn that ∠*B*+∠*C*< ∠*CAX*; which is absurd. Therefore, *DA*=*DB*, or *DC*.

Next, let the vertical ∠*CEB*, of the △*EBC*, be acute, and let *ED* join *E* and the bisection, *D*, of *BC*,* *ED*>*BD*, or *DC*.

Suppose ∠*C* to be that which is not the less of the two angles *B*, *C*: from *C* draw (E. XII. 1.) *CA* perpendicular to the opposite side *EB*; and join *A*, *D*: Then (Supp. VII. 1.) *CA* falls within *EB*; and, since (*constr.*) the ∠ *CAE* is a right angle, the ∠ *DAE* is greater than a right angle; therefore (E. XVII. 1.)

the $\angle$ ***AED*** is less than a right angle; and therefore less than the $\angle$ ***DAE***; therefore (E. XIX. 1.) ***DE*** > ***DA***; but, by the former case, ***DA***=***DB***; therefore ***DE***>***DB***, or ***DC***.

Lastly, if ***FBC*** be an obtuse-angled triangle, obtuse-angled at ***F***, join ***F***, ***D***; draw, as before, ***CA*** perpendicular to ***BF***; and join ***A***, ***D***: Then (Supp. VII. 1.) ***CA*** falls without ***BF***, and the $\angle$***AFD*** (E. XVI. 1.) >$\angle$***FBD***; but since (1st case) ***DA***= ***DB***, the $\angle$ ***DBF***= $\angle$ ***DAF*** (E. V. 1.); ∴ $\angle$ ***AFD*** >$\angle$ ***DAF***; (E. XIX. 1.) ***DA*** > ***DF***; but ***DA***=***DB***; ∴ ***DF***< ***DB***, or ***DC***.

Or, the two last cases may be proved, ex absurdo, in the same manner as the first is proved.

40. Cor. 1. If any number of triangles have a right angle for their common vertical angle, and have equal hypotenuses, the locus of the bisections of the several hypotenuses is a quadrantal arch of a circle, having the common vertex for its centre, and the half of any hypotenuse for its radius.

For, the bisections of the hypotenuses will, each of them, (Supp. XXIX. 1.) be at a distance from the common vertex equal to the half of one of the equal hypotenuses; *i. e.* they will all be at distances from that point, equal to the half of any one of those equal lines: It is manifest, therefore, that they will be in the circumference of a circle, described from that point as the centre, at a distance equal to the half of one of the hypotenuses; and the portion of the circumference, being that which is intercepted by two radii at right angles to each other, is a quadrantal arch.

41. Cor. 2. A circle described from the bisection of the hypotenuse of a right-angled triangle as a

centre, at the distance of half the hypotenuse, will pass through the summit of the right angle.

42. Cor. 3. The vertical angle of a triangle being a right angle, a point in the base, which is equidistant from the vertex and from either extremity of the base, bisects the base.

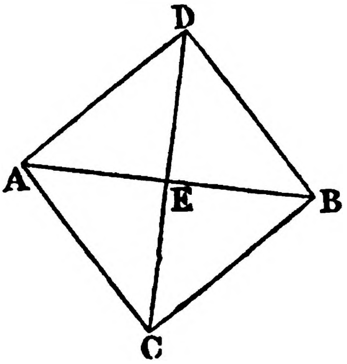
Let the point *D*, in the base *BC* of the △ *ABC*, having the ∠ *B* a right angle, be equidistant from either extremity, as *B*, of *BC*, and from the angular point *A*: The point *D* bisects *BC*.

For, if not, let *G* be the bisection of *BC*, and join *D*, *A* and *E*, *A*: Then, since, (*hyp.*) $DA = DB$; therefore (E. v. 1.) the $\angle DAB = \angle DBA$: also, since *G* is the bisection of *BC*, therefore (Sect. xxix. 1.) $GA = GB$; therefore (E. v. 1.) the $\angle GAB = \angle GBA$; therefore the $\angle GAB = \angle DAB$, the greater to the less, which is absurd; therefore no other point than *D* can be the bisection of *BC*.

Prop. XXX.

43. Problem. *Upon a given finite straight line, as a* diameter, *to describe a square.*

Let *AB* be a finite straight line: Upon *AB*,



as a diameter, it is required to describe a square.

Bisect (E. x. 1.) *AB* in *E*; through *E* draw (E. xi. 1.) *DEC* perpendicular to *AB*, and make (E. iii. 1.) *ED* and *EC* each of them equal to *AE* or *EB*: Join *A*, *D*, and *D*, *B*, and *B*, *C*, and *C*, *A*: The figure *ADBC* is a square, having *AB* for its diameter.

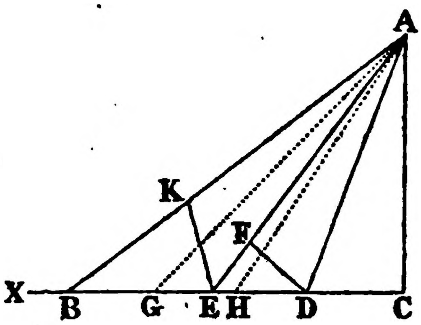
For since (*constr.*) *DE* = *EC*, and *AE* is common to the △s *AED*, *AEC*, and that the right ∠ *AED* = right ∠ *AEC*; therefore (E. iv. 1.) *AD* = *AC*; and in the same manner *AD* may be shewn to be equal to *DB*, and *DB* to *BC*; therefore the figure *ADBC* is equilateral.

Again, since (*constr.*) *AE* = *DE*, the ∠ *EAD* = ∠ *EDA* (E. v. 1.); but (*constr.*) ∠ *AED* is a right angle; therefore each of the ∠s *EAD*, *EDA*, is half a right angle; and, in the same manner, may each of the ∠s *EDB*, *DBE*, *CBE*, *BCE*, *ECA*, *EAC*, be shewn to be half a right angle; therefore all the angles of the figure *ADBC* are right angles; and it has been proved that all its sides are equal; therefore (E. xxx. Def. 1.) *ADBC* is a square.

PROP. XXXI.

44. THEOREM. *If either of the acute angles of a given right-angled triangle be divided into any number of equal angles, then, of the segments of the base, subtending those equal angles, the nearest to the right angle is the least; and, of the rest, that which is nearer to the right angle is less than that which is more remote.*

Let there be a right-angled triangle, right-angled at *C*, and let the acute angle at *A* be divided into any



number of equal angles, *CAD*, *DAE*, *EAB*, &c.; then is *CD* the least of the segments of the base subtending those equal angles, and of the rest $DE<EB$; and so on.

For, at the point *D* in *AD* make (E. XXIII. 1.) the $\angle ADF=\angle ADC$: And since, also, the $\angle CAD=\angle DAE$ (*hyp.*) and *AD* common to the two $\triangle$s *ACD*, *AFD*; therefore (E. XXVI. 1.) $DF=DC$: But (E. XIX. 1. and E. XXXII. 1.) $DE>DF$; $\therefore DE>DC$; *i. e.* $DC<DE$.

Again, at the point *E*, in *AE*, make the $\angle AEK=\angle AED$; and it may, in like manner, be shewn that $EK=ED$: But (E. XVI. 1.) $\angle BKE>\angle AEK$; $\therefore \angle BKE>\angle AED$; and $\angle AED>\angle ABE$; much more, then, is $\angle BKE>\angle EBK$; therefore (E. XIX. 1.) $BE>EK$ or *ED*; *i. e.* $ED<EB$.

And in the same manner may *EB* be shewn to be less than the next segment that is more remote from *C*; and so on.

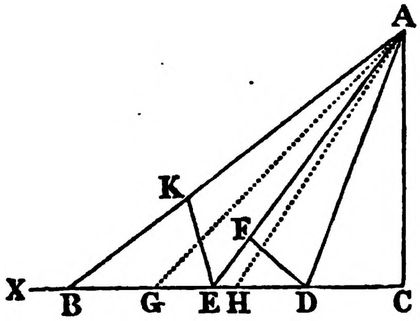
45. Cor. It is manifest, from the demonstration, that if any three straight lines, *AB*, *AE*, *AD*, be drawn to the given straight line *XC* from a given point *A*, without it, so that the $\angle BAE=\angle EAD$,

the segment BE, of XC, which is the further from the perpendicular AC, shall be greater than the segment ED, which is the nearer to AC.

PROP. XXXII.

46. THEOREM. *If either angle at the base of a triangle be a right angle, and if the base be divided into any number of equal parts, that which is adjacent to the right angle shall subtend the greatest angle at the vertex; and, of the rest, that which is nearer to the right angle shall subtend, at the vertex, a greater angle than that which is more remote.*

Let ACB be a right-angled triangle, right-angled at C, and let the base BC be divided into any number of equal parts CD, DH, HG, &c.: Of these segments DC shall subtend the greatest angle at the vertex A; and of the rest DH shall subtend, at A, a greater angle than HG; and so on.



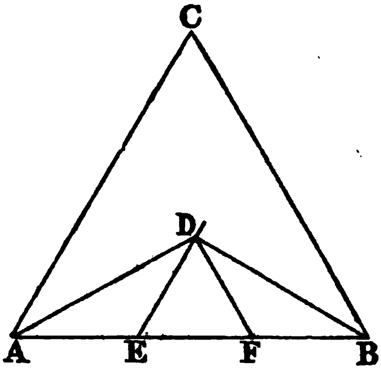
For, join A, D, and A, H, and A, G, &c.; also, at the point A, in DA, make (E. XXIII. 1.) the $\angle DAE = CAD$: Then (Supp. xxxi. 1.) $ED > DC$; but (*hyp.*) $DC = DH$; $\therefore ED > HD$, and it is manifest that the $\angle EAD > \angle HAD$; but (*constr.*) $\angle EAD = \angle CAD$; $\therefore$ $\angle CAD > \angle DAH$:

And, in the same manner, it may be shewn, by the help of the corollary to Supp. xxxi. 1. that the $\angle DAH > \angle HAG$; and so on.

PROP. XXXIII.

47. PROBLEM. *To trisect a given finite straight line.*

Let *AB* be the given straight line: It is required to divide it into three equal parts.



Upon *AB* describe (E. I. 1.) the equilateral triangle *CAB*; bisect (E. IX. 1.) the two equal $\angle$s *A* and *B*, by the straight lines *AD* and *BD*, which meet in *D*; and from *D* draw (E. XXXI. 1.) *DE* parallel to *CA*; and *DF* parallel to *CB*: Then are *AE*, *EF* and *FB* equal to one another.

For, since (E. XXIX. 1. and *constr.*) the $\angle DEF = \angle CAB$, and $\angle DFE = \angle CBA$; therefore (Supp. xxvi. 1.) $\angle EDF = \angle ACB$; but (E. V. 1. *Cor.* and *constr.*) the $\triangle CAB$ is equiangular; therefore the $\triangle DEF$

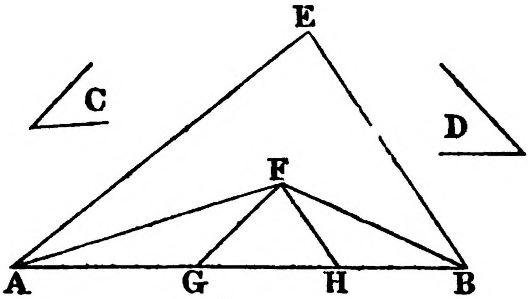
is equiangular; and therefore (E. VI. 1. *Cor.*) it is, also, equilateral; so that *DE* and *DF* are, each of them, equal to *EF*.

Again, since (E. XXIX. 1. and *constr.*) the ∠*EDA* = ∠*DAC*; and that (*constr.*) the ∠*DAC* = ∠*DAE*; ∴ ∠*EDA* = ∠*DAE*; therefore (E. VI. 1.) *AE* = *DE*; but *DE* has been proved to be equal to *EF*; ∴ *AE* = *EF*; and in the same manner, *EF* may be shewn to be equal to *FB*; therefore *AB* has been divided into three equal parts *AE*, *EF*, and *FB*.

PROP. XXXIV.

48. **PROBLEM.** *To describe a triangle which shall have its three sides, taken together, equal to a given finite straight line, and its three angles equal to three given angles, each to each; the three given angles being together equal to two right angles.*

Let *AB* be a finite straight line, and *C* and *D* two rectilineal angles: It is required to describe a triangle, which shall have its perimeter equal to *AB*, two of its angles equal to *C* and *D*, each to each, and its third angle equal to an angle, which, together with *C* and *D*, makes up two right angles.



At the point *A*, in *AB*, make (E. XXIII. 1.) the $\angle BAE = \angle C$; and at the point *B* make the $\angle ABE = \angle D$; therefore (Supp. xxvi. 1.) the $\angle AEB$ is equal to the third angle of the triangle which is to be described: Bisect (E. IX. 1.) the $\angle$s *EAB*, *EBA*, by *AF* and *BF*, which meet in *F*; and through *F* draw (E. XXXI. 1.) *FG* parallel to *EA*, and *FH* parallel to *EB*: Then is *FGH* the triangle, which was to be described.

For, since (*constr.*) *FG* is parallel to *EA*, and *FA* meets them, therefore (E. XXIX. 1.) the $\angle EAF = \angle AFG$; but (*constr.*) the $\angle EAF = \angle FAG$; therefore the $\angle FAG = \angle AFG$; therefore (E. VI. 1.) $FG = GA$; and, in the same manner, it may be shewn that $FH = HB$; $\therefore FG + GH + HF = AG + GH + HB$; *i. e.* the perimeter of the $\triangle FGH$ is equal to the given straight line *AB*.

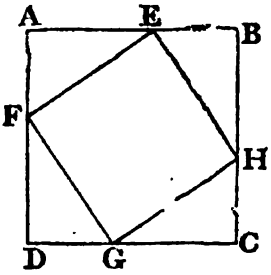
Again, because *FG* is parallel to *EA*, and *FH* is parallel to *EB*, therefore (E. XXIX. 1.) the $\angle FGH = \angle EAB$, and $\angle FHG = \angle EBA$; but (*constr.*) the $\angle EAB = \angle C$, and the $\angle EBA = \angle D$; therefore also, the $\angle FGH = \angle C$, and the $\angle FHB = \angle D$; therefore (Supp. xxvi. 1.) the $\angle GFH$ is equal to the third angle of the triangle, which was to be described; therefore the $\triangle FGH$, the perimeter of which has been shewn to be equal to the given straight line *AB*, is the triangle which was to be described.

PROP. XXXV.

49. THEOREM. *If, in the sides of a given square, at equal distances from the four angular points, four other points be taken, one in each side, the*

figure contained by the straight lines which join them, shall also be a square.

Let *ABCD* be a square; in the sides *AB*, *BC*, *CD*, *DA*, let the four points *E*, *H*, *G*, *F* be taken, so that *E* is at the same distance from *A* that *H* is from *B*, that *G* is from *C*, and *F* from *D*; and let *E*, *H*, and *H*, *G*, and *G*, *F*, and *F*, *E*, be joined: The figure *EFGH* is a square.



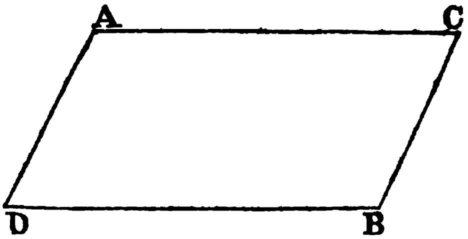
For, since (E. xxx. Def. 1.) all the sides of the given square *ABCD* are equal, and that (*hyp.*) $AE = BH = DF$, it is manifest that the two triangles *FAE*, *EBH* have the two sides *FA*, *AE* equal to the two *EB*, *BH*, each to each, and (E. xxx. Def. 1.) the $\angle A = \angle B$; therefore (E. IV. 1.) the $\angle AFE = \angle BEH$; and $FE = EH$: And, in the same manner, it may be shewn that $EH = HG = GF$; therefore the figure *EFGH* is equilateral.

Again, since, as hath been proved, the $\angle AFE = \angle BEH$; therefore the $\angle AFE + \angle AEF = \angle BEH + \angle AEF$; but, since the $\angle A$ is a right angle, therefore (E. XXXII. 1.) $\angle AFE + \angle AEF =$ a right angle; therefore also, $\angle BEH + \angle AEF =$ a right angle; but (E. xv. 1. Cor. 2.) $\angle BEH + \angle AEF + \angle HEF =$ two right angles; therefore the $\angle HEF$ is a right angle; and, in the same manner, may the remaining angles of the figure *EFGH,* which has been shewn to be equilateral, be proved to be right angles; therefore (E. xxx. Def. 1.) *EFGH* is a square.

PROP. XXXVI.

50. THEOREM. *If the opposite angles, of a quadrilateral figure be equal to each other, the figure shall be a parallelogram.*

Let *AB* be a quadrilateral figure, having the angle *A* equal to the opposite angle *B,* and the angle *C* to the opposite angle *D*: The figure *ADBC* is a parallelogram.



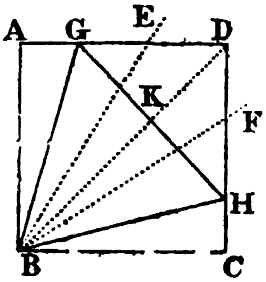
For, (E. xxxii. 1. Cor. 1.) the four angles of the figure *ADBC* are together equal to four right angles; and, by the hypothesis, the four angles are the double of the two angles, *DAC, ACB*; it is manifest, therefore, that the two ∠s *DAC, ACB* are together equal to two right angles; therefore (E. xxviii. 1.) *AD* is parallel to *CB*: And, in the same manner, *AC* may be shewn to be parallel to *DB*; therefore the figure *ADBC* is a parallelogram.

PROP. XXXVII.

51. PROBLEM. *In a given square to inscribe an equilateral triangle, having one of its angular*

points upon one of the angular points of the square, and its two remaining angular points one in each of two adjacent sides of the square.

Let *ABCD* be a square: It is required to inscribe in it an equilateral triangle, having one of its angular points upon the angular point *B* of the square.



Trisect (Supp. xx. 1.) the right ∠*ABC*, by *BE* and *BF*; bisect (E. IX. 1.) the ∠s *ABE*, *CBF* by *BG* and *BH*, meeting *AD* and *DC* in *G* and *H*, respectively; and join *G*, *H*: The △*GBH* is equilateral.

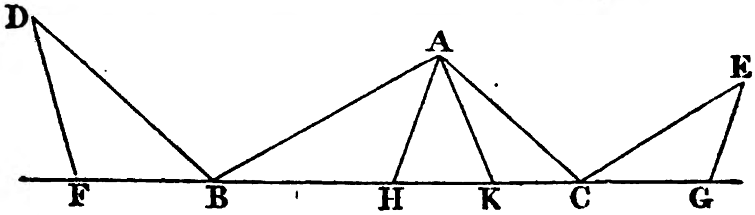
For, join *B*, *D*, and let *BD* meet *GH* in *K*: Then, it is manifest from the construction, that the ∠*ABG*= ∠*CBH*; also, (*hyp.* and E. XXX. Def. 1.) the ∠*A*= ∠*C*, and the side *AB*, of the △*ABG*, is equal to the side *CB*, of the △*CBH*; therefore (E. XXVI. 1.) *BG*=*BH*; therefore (E. V. 1.) the ∠*BGH*= ∠*BHG*; also, (*constr.* and E. VIII. 1.) the ∠*GBD*= ∠*HBD*; and *BK* is common to the two △s *BKG*, *BKH*; therefore (E. XXVI. 1.) the ∠*BKG* = ∠*BKH*; therefore (E. X. Def. 1.) each of these angles is a right angle; therefore (E. XXXII. 1.) ∠*KGB*+ ∠*GBK*=a right ∠= ∠*GBH*+2∠*ABG*

$= \angle GBH + \angle ABE$ (*constr.*); but, since (*constr.*) $\angle ABG + \angle CBH = \angle EBF$, add to each of these equals the $\angle$s EBG, FBH, and the $\angle GBH = 2\angle ABE$; therefore the $\angle GBK = \angle ABE$; and it has been shewn that $\angle KGB + \angle GBK = \angle GBH + \angle ABE$; $\therefore \angle KGB = \angle GBH$; $\therefore HG = GB = BH$; *i. e.* the $\triangle GBH$ is equilateral.

PROP. XXXVIII.

52. THEOREM. *If, at the extremities of the base of a given triangle, two straight lines be drawn, both above the base, and each of them equal to the adjacent side, and making with it an angle equal to the vertical angle of the triangle; then, if two straight lines, let fall from the extremities of the two so drawn, make, with the base produced, two angles that are equal each of them to the vertical angle, they shall cut off equal segments from the base produced.*

From the extremities B, C, of the base BC of the given $\triangle ABC$, let BD be drawn equal to the adjacent side AB, and CE equal to the adjacent side AC, making the $\angle$s ABD, ACE, each equal to the vertical $\angle BAC$ of the triangle, and let DF and EG, drawn from D and E, make with BC produced



the $\angle$s DFB, EGC, each also equal to the $\angle BAC$: Then shall $FB = GC$.

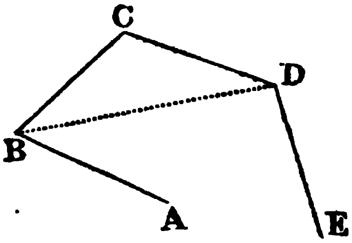
For, from the point A draw (Supp. xxv. 1.) AH and AK making with BC the $\angle$s AHB, AKC each equal to the $\angle DFB$, or BAC, or CGE: And, since (E. XIII. 1.) $\angle ABH + \angle ABD + \angle DBF =$ two right angles $= \angle DBF + \angle BFD + \angle FDB$ (E. XXXII. 1.), and that (*constr.*) $\angle ABD = \angle BFD$; $\therefore \angle ABH = \angle FDB$; but, (*constr.*) $\angle AHB = \angle DFB$, and the side AB of the $\triangle AHB$ is equal to the side DB of the $\triangle DFB$; therefore (E. XXVI. 1.) $FB = AH$: And in the same manner GC may be shewn to be equal to AK; but since (*constr.*) the $\angle AHB = \angle AKC$, therefore (E. XIII. 1.) the $\angle AHK = \angle AKH$; therefore (E. VI. 1.) $AH = AK$; and FB was shewn to be equal to AH, and GC to AK; $\therefore FB = GC$.

53. COR. If the vertical $\angle BAC$ be a right angle, the two straight lines AH and AK coincide; and the segments FB, GC are equal each of them to the perpendicular drawn from A to the base BC: In this case, also, $DF = BK$, and $EG = CK$.

PROP. XXXIX.

54. THEOREM. *If four straight lines cut each other, without including space, but so as to make three internal angles, towards the same parts, which together are less than four right angles, the two lines, which are not joined, shall meet, if produced far enough.*

Let the four straight lines AB, BC, CD, DE, cut one another, without enclosing space, so that the ∠s *ABC, BCD, CDE,* are together less than four right angles; Then shall *BA* and *DE* meet, if they are produced far enough.

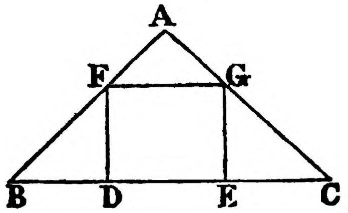


For, join *B, D*: And (E. xxxii. 1.) ∠*DBC*+ ∠*BCD*+*CDB* = two right angles; if, therefore, these three angles be taken from the three given angles, which (*hyp.*) are less than four right angles; there will remain the two ∠s *ABD, EDB,* together less than two right angles; therefore (E. xii. Axiom 1.) *BA* and *DE* will meet if they be continually produced.

Prop. XL.

55. Problem. *To inscribe a square in a given right-angled isosceles triangle.*

Let *ABC* be the given isosceles triangle, right-angled at *A*: It is required to inscribe a square in the △*ABC*.



Trisect (Supp. xxxiii. 1.) the hypotenuse *AC,* in the points *D* and *E*; from *D* and *E* draw (E. XI. 1.) *DF* and *EG* perpendicular to *BC*, meeting the sides *AB* and *AC* in *F*, and *G*, respectively; and join *F*, *G*: The inscribed figure *FDEG* is a square.

For, since the $\angle A$ is a right angle, and that (*hyp.* and E. V. 1.) $\angle B = \angle C$; therefore (E. XXXII. 1.) $\angle B$ is half a right angle; but (*constr.*) the $\angle D$ is a right angle; therefore the $\angle DFB$ is half a right angle, and is, therefore, equal to the $\angle FBD$; therefore (E. VI. 1.) $DF = BD$; but (*constr.*) $BD = DE$; $\therefore DF = DE$; and, in the same manner, it may be shewn that $EG = DE$; $\therefore DF = DE = EG$.

Again, since (*constr.*) the $\angle$s *D* and *E* are right angles, therefore (E. XXVIII. 1.) *DF* is parallel to *EG*; and it has been shewn that $DF = EG$; therefore (E. XXXIII. 1.) *FG* is equal and parallel to *DE*; therefore (E. XXIX. 1.) the figure *FDEG* has all its angles right angles; and it is equilateral; therefore (E. XXX. Def. 1.) it is a square.

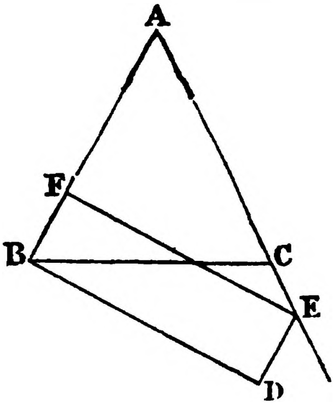
Prop. XLI.

56. Problem. *To find a point, in either of the equal sides of a given isosceles triangle, from which, if a straight line be drawn, perpendicular to that side, so as to meet the other side produced, it shall be equal to the base of the triangle.*

Let *ABC* be the given isosceles triangle: It is required to find, in either of the two equal sides, as *AB*, a point from which if a perpendicular be drawn to *AB* and produced to meet *AC*, produced, it shall be equal to the base *BC*.

Draw (E. XI. 1.) from *B*, *BD* perpendicular to

AB, and make (E. III. 1.) *BD* = *BC*; from *D* draw



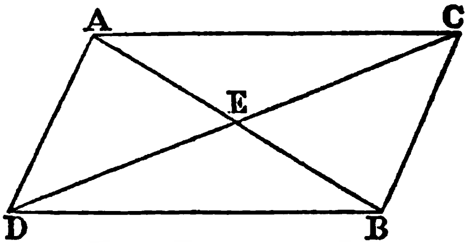
(E. XXXI. 1.) *DE* parallel to *AB*, meeting *AC* produced in *E*; and from *E*, draw *EF* parallel to *BD*: *F* is the point which was to be found.

For (*constr.*) the figure *FBDE* is a parallelogram; therefore (E. XXXIV. 1.) *FE* = *BD* = *BC* (*constr.*); also, since (*constr.*) the ∠*FBD* is a right angle, the ∠*BFE* is, also, (E. XXIX. 1.) a right angle.

Prop. XLII.

57. Theorem. *The diameters of a parallelogram bisect each other.*

Let *AB* and *CD* be the diameters of the parallelogram *ADBC*; *AB* and *CD* bisect one another in the point of their intersection *E*.

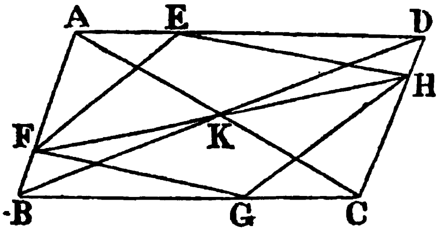


For since *ADBC* is a parallelogram, *AD*=*CB* (E. XXXIV. 1.) and (E. XXIX. 1.) the ∠*EAD* of the △*AED*, =∠*EBC*, of the △*BEC*, and the ∠*EDA*=∠*ECB*; therefore (E. XXVI. 1.) *AE*=*EB*, and *DE*=*EC*.

PROP. XLIII.

58. THEOREM. *If in two opposite sides of a parallelogram two points be assumed, one in each of those sides, equidistant from two opposite angles of the figure, and if two other points be likewise assumed, in the two other opposite sides, equidistant from the same two angles, the figure, contained by the straight lines joining the four points so assumed, shall be a parallelogram.*

In the opposite sides *AD*, *BC* of the ▭ *ABCD*,



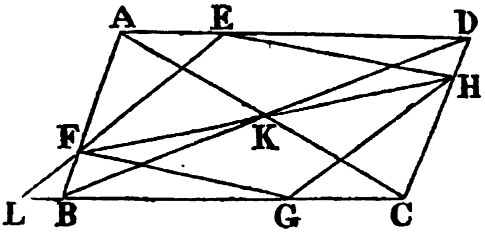
let the points *E* and *G* be taken equidistant from the opposite ∠s *A* and *C*; let also, the points *F* and *H* be taken, in the other two opposite sides, *AB* and *DC*, equidistant from *A* and *C*; and let *E*, *F*, and *F*, *G*, and *G*, *H*, and *H*, *E*, be joined: The figure *EFGH* is a parallelogram.

For since (*hyp.*) *AE*=*CG*, and *AF*=*CH*, and that (E. XXXIV. 1.) the ∠*A*=∠*C*; therefore (E. IV. 1.) *FE*=*GH*: Again since (E. XXXIV. 1.) *AB*=*DC*, and *AD*=*BC*, and that (*hyp.*) of *AD* the part *AE* is equal to the part CG of BC, and of AB the part AF is equal to the part CH of DC; $\therefore ED = GB$, and $DH = BF$; also (E. XXXIV. 1.) the $\angle EDH = \angle FBG$; therefore (E. IV. 1.) $EH = FG$; and it has been proved that $EF = HG$; therefore (Supp. xviii. 1.) $EFGH$ is a parallelogram.

PROP. XLIV.

59. THEOREM. *If any number of parallelograms be inscribed in a given parallelogram, the diameters of all the figures shall cut one another in the same point.*

Let $ABCD$ be the given parallelogram, and let



$EFGH$ be any parallelogram, inscribed in $ABCD$: The diameters of $ABCD$ and of $EFGH$ cut one another in the same point.

For draw AC a diameter of $ABCD$, and FH a diameter of $EFGH$; let AC and FH cut one another in K; and let CB, produced, meet EF, produced, in L: Then, since AE is parallel to BC, and EF parallel to HG, the $\angle CGH =$ (E. XXIX. 1.) $\angle GLE$; and the $\angle GLE = \angle LEA$; therefore the $\angle CGH = \angle AEF$; also (*hyp.* and E. XXXIV. 1.) the $\angle A = \angle C$, and the side $FE =$ the opposite side GH, of the ▭EFG; therefore (E. XXVI. 1.) $CH = AF$:

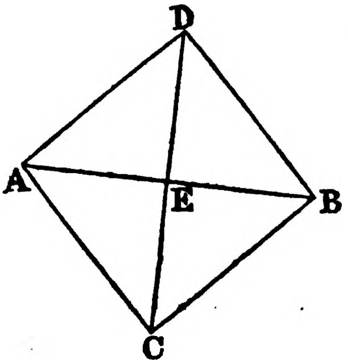
Again, since the side AF of the $\triangle AKF =$ the side CH of the $\triangle CKH$, and that (E. xxix. 1.) the $\angle$s KAF, KFA are equal to the $\angle$s KCH, KHC; therefore (E. xxvi. 1.) $AK = KC$, and $FK = KH$; *i. e.* K is the bisection of the diameters AC, FH; therefore (Supp. xlii. 1.) all the diameters cut one another in the point K.

60. Cor. From the demonstration it is manifest, that the angle contained by any two given straight lines, is equal to the angle contained by two other straight lines, that are parallel to the two given straight lines, each to each.

Prop. XLV.

61. Theorem. *The diameters of an equilateral four-sided plane rectilineal figure bisect one another at right angles.*

Let AB and DC be the diameters of the equilateral four-sided figure $ACBD$, cutting one another



in E: AB and DC bisect one another in E, at right angles.

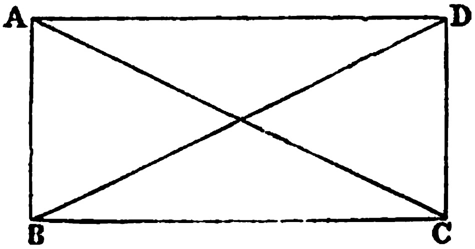
For, since (*hyp.*) $ACBD$ is equilateral, it is (Supp. xviii. 1.) a parallelogram; and therefore (Supp. xlii. 1.)

the diameters bisect one another in *E*: Again, because *DE* = *CE*, and *EA* is common to the two △s *AED*, *AEC*, and that (*hyp.*) *AD* = *AC*; therefore (E. VIII. 1.) the ∠*AED* = ∠*AEC*; *i. e.* (E. X. Def. 1.) each of the ∠s *AED*, *AEC* is a right angle; therefore (E. XV. 1.) each of the ∠s *DEB*, *CEB*, is, also, a right angle.

PROP. XLVI.

62. THEOREM. *The diameters of a rectangle are equal to one another.*

Let *AC*, and *BD* be the diameters of the rectangle *ABCD*: Then *AC* = *BD*.



For, since (*hyp.*) the opposite angles of the figure are equal, each being a right angle; therefore (Supp. XXVI. 1.) the figure *ABCD* is a parallelogram; therefore (E. XXXIV. 1.) *AD* = *BC*; and *AB* is common to the two △s *ABC*, *BAD*, and the ∠*ABC* = ∠*BAD*; therefore (E. IV. 1.) *AC* = *BD*.

PROP. XLVII.

63. PROBLEM. *To inscribe a square in a given equilateral four-sided figure.*

Let $ABCD$ be the given equilateral four-sided

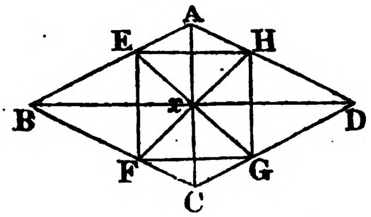


figure: It is required to inscribe a square in $ABCD$.

Join A, C, and B, D, and let $\overline{AC}$ and $\overline{BD}$ cut one another in X; bisect (E. ix. 1. and E. xv. 1.) the ∠s AXB and CXD, by the straight line EG, and the ∠s BHC, AXD, by the straight line FH; and join E, F, and F, G, and G, H, and H, E: The inscribed figure $EFGH$ is a square.

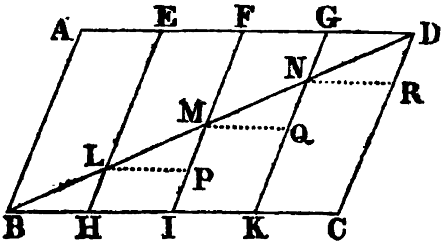
For, since the figure $ABCD$ is (*hyp.*) equilateral, AC and BD (Supp. xlv. 1.) bisect one another at right angles; therefore (*constr.*) each of the ∠s EXA, AXH, HXD, DXG, GXC, CXF, FXB, BXE, is half a right angle; therefore the ∠s EXH, HXG, GXF, FXE are right angles: Again, because $ABCD$ is equilateral, it is (Supp. xviii. 1.) a parallelogram; therefore (E. xxviii. 1.) the $\angle BAC = \angle ACD$; but, because (*hyp.*) $\overline{DC} = \overline{DA}$; therefore (E. v. 1.) the $\angle ACD = \angle DAC$; therefore the $\angle BAC = \angle DAC$; *i. e.* the $\angle EAX = \angle HAX$; and it has been shewn that the $\angle EXA = \angle HXA$; and $\overline{AX}$ is common to the two △s AEX, AHX; therefore (E. xxvi. 1.) $\overline{EX} = \overline{HX}$; and, in the same manner it may be shewn that EX, HX, GX, and FX, are all equal: and the angles contained by those lines are equal, being right angles; therefore (E. iv. 1.) the figure $EFGH$ is equilateral, and therefore (Supp. xviii. 1.)

it is a parallelogram; and since the $\angle EXH$ is a right angle, and that $XE=XH$; therefore (E. v. 1. and E. xxxii. 1.) each of the angles XEH, XHE is half a right angle: In the same manner it may be shewn that each of the $\angle$s XHG, XGH, XGF, XFG, XFE, XEF is half a right angle; therefore the figure $EFGH$, which has been shewn to be equilateral, has all its angles right angles; therefore (E. xxx. Def. 1.) it is a square.

Prop. XLVIII.

64. **Theorem.** *If two opposite sides of a parallelogram be divided each into the same number of equal parts, the straight lines, joining the opposite points of division, shall also divide the diameter of the parallelogram into the same number of equal parts.*

Let the two opposite sides AD, BC, of the ▱$ABCD$, of which BD is a diameter, be divided



into any number of equal parts, AE, EF, FG, &c., BH, HI, IK, &c.; and let E, H, and F, I, and G, K, &c. be joined: The diameter BD is divided, by EH, FI, GK, &c. into the same number of equal parts, BL, LM, MN, &c. as either of the opposite sides AD or BC.

For, through L, M, N, &c. draw (E. xxxi. 1.) LP, MQ, NR, &c. each parallel to AD or BC:

Then, since *AE* is equal and parallel to *BH*, *EH* (E. XXXIII. 1.) is parallel to *AB*; and in the same manner it may be shewn, that *FI*, *GK*, &c., are parallel to one another; therefore the figures *LI*, *MK*, *NC*, &c., are parallelograms; therefore (E. XXXIV. 1.) $LP = HI$; but (*hyp.*) $HI = BH$; therefore, $BH = LP$; and since *LP* is parallel to *BC*, and *LH* parallel to *MI*, and that $\overline{MLB}$ meets these parallels; therefore (E. XXIX. 1.) the ∠s *HBL*, *BLH*, of the △*BLH*, are equal to the ∠s *PLM*, *LMP*, of the △*LMP*; and it has been proved that the side $BH = LP$; therefore (E. XXVI. 1.) $BL = LM$: And, in the same manner it may be shewn, that $LM = MN$; $MN = ND$, and so on.

65. COR. From the demonstration it is manifest that if the one of two given straight lines, or a part of it, be divided into any number of equal parts, and from the points of division parallel straight lines be drawn cutting the other, the segments of that other given line, between these parallels, will be equal to one another.

PROP. XLIX.

66. PROBLEM. *To divide a given finite straight line into any given number of equal parts.*

Let *BD* be a finite straight line (See fig. p. 50.): It is required to divide it into any given number of equal parts.

From *B* draw an indefinite straight line *BC* making any angle with *DB*; and from *D* draw (E. XXXI. 1.) *DA*, also indefinite, and parallel to *BC*; take any point *H* in *BC*; make (E. III. 1.) *HI*, *IK*, *DG, GF, FE,* each equal to *BH,* so that the number of these equal straight lines in *BC,* and also in *DA,* may be less by one than the given number of parts, into which *BD* is to be divided; and join *E, H,* and *F, I* and *G, K*: The straight lines *EH, FI,* and *GK,* will divide *BD* into the required number of equal parts.

For, in *BC,* and *DA,* take *KC,* and *EA,* each equal to *BH,* and join *A, B* and *D, C*: Then, since (*constr.*) *AD* is equal and parallel to *BC, AB* is also (E. XXXIII. 1.) parallel to *DC*; ∴ *ABCD* is a parallelogram, of which *BD* is a diameter; therefore (*constr.* and Supp. xlviii. 1.) *BD* is divided by *EH, FI,* and *GK,* into as many equal parts as *BC,* or *AD,* is divided into.

Otherwise,

Draw *BC,* as before, and make the number of equal parts *BH, HI, IK, KC,* equal to the given number into which *BD* is to be divided; join *C, D*; and draw *HL, IM, KN,* each parallel to *CD*: Then will these parallels divide *BD* into the required number of equal parts.

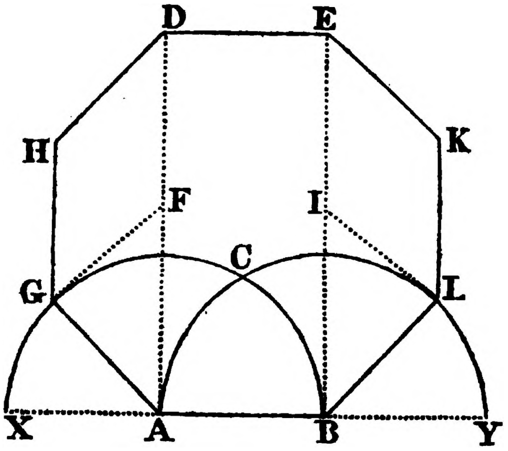
For, if *LP, MQ, NR* be drawn each parallel to *BC,* it may be proved, (as in Supp. xlviii. 1.) that $BL = LM = MN = ND$.

PROP. L.

67. PROBLEM. *Upon a given finite straight line to describe an equilateral and equiangular octagon.*

Let *AB* be a finite straight line: Upon *AB,* it is required to describe an equilateral and equiangular octagon.

From the points *A* and *B* draw (E. XI. 1.) *AD* and *BE* perpendicular to *AB*, and produce *AB* both



ways to *X* and *Y*; bisect (E. IX. 1.) the ∠s *DAX*, *EBY*, by *AG*, and *BL*, and make *AG* and *BL*, each equal to *AB*; from the points *G* and *L*, draw *GF* perpendicular to *AG*, and *LI* perpendicular to *BL*; also, draw (E. XXXI. 1.) *GH* parallel to *AD*, and make *GH*=*AB* or *AG*; in like manner, draw *LK* parallel to *BE*, and make *LK*=*AB*; lastly, draw *HD* parallel to *GF*, meeting *AD* in *D*, and *KE* parallel to *LI*, meeting *BE* in *E*; and join *D*, *E*: The figure *ABLKEDHG*, described on *AB*, is an equilateral and equiangular octagon.

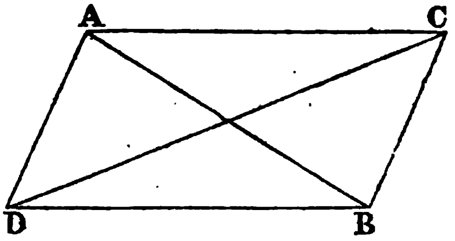
For, since (*constr.*) the side *AG*, of the △ *AGF*, is equal to the side *BL*, of the △ *BLI*, and ∠ *GAF* = ∠ *LBI*, and that the ∠s *AGF*, *BLI* are equal, being right angles; therefore, (E. XXVI.) *AF*=*BI*; also, since (*constr.*) *HF* and *KI* are parallelograms, *FD*=*GH*, and *IE*=*LK*; but (*constr.*) *GH*=*LK*; ∴ *FD*=*IE*; therefore the whole *AD*=the whole *BE*; and (*constr.* and E. XXVIII. 1.) *AD* is parallel to *BE*; therefore (E. XXXIII. 1.) *DE* is equal and parallel to *AB*; and therefore (*constr.* and E. xxxiv. 1.) the ∠s *ADE, BED* are right angles: Again, since (*constr.*) the ∠s *AGF, BLI*, are right angles, and that the ∠s *GAF, IBL* are each the half of a right angle; therefore (E. xxxii. 1.) the ∠s *GFA, LIB*, are each the half of a right angle; $\therefore AG = GF = BL = LI$; and (E. xxxiv. 1.) $HD = GF$, and $KE = LI$; whence it is manifest that the figure *ABLKEDHG* is equilateral.

Lastly, since *HG* is parallel to *DA*, and *KL* to *EB*, and *FG* and *IL* meet these parallels; therefore (E. xxix. 1.) the ∠ *HGF* = ∠ *GFA*, and ∴ ∠ *HGF* = the half of a right angle; therefore (E. xxxiv. 1.) the ∠ *HDF* is the half of a right angle; in the same manner, it may be shewn that each of the ∠s *IEK, KLI*, is the half of a right angle; and it has been proved that the ∠s *ADE, BED* are right angles; whence, and from the construction, it is manifest, that the figure *ABLKEDHG*, which has been shewn to be equilateral, is also equiangular.

PROP. LI.

68. THEOREM. *If either diameter of a parallelogram be equal to a side of the figure, the other diameter shall be greater than any side of the figure.*

Let the diameter *AB*, of the ▭ *ACBD*, be equal



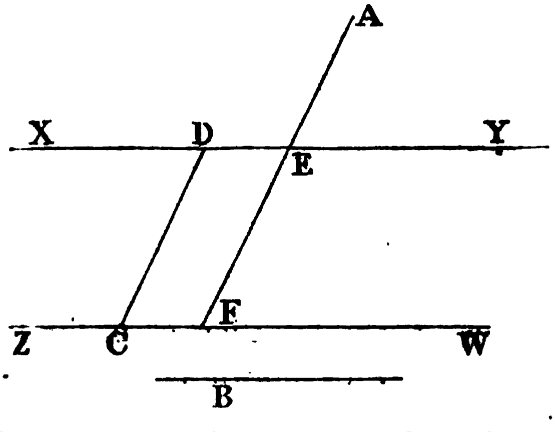
to the side *AC*: The other diameter *CD* shall be greater than either *AC* or *AD*.

For, because $AC = AB$, the $\angle ACB = \angle ABC$ (E. v. 1.) and (*hyp.* and E. xxix. 1.) the $\angle DAB = \angle ABC$; but the $\angle DAC > \angle DAB$; therefore the $\angle DAC > \angle ACB$; and the sides DA, AC, of the $\triangle DAC$, are (E. xxxiv. 1.) equal to the sides BC, CA, of the $\triangle BCA$; therefore (E. xxiv. 1.) $CD > AB$; but (*hyp.*) $AB = AC$; $\therefore CD > AC$: And it has been shewn that the $\angle DAC > \angle ACB$; much more then is the $\angle DAC > \angle ACD$; therefore (E. xix. 1.) $DC > AD$.

Prop. LII.

69. Problem. *From a given point to draw a straight line cutting two parallel straight lines, so that the part of it, intercepted between them, shall be equal to a given finite straight line, not less than the perpendicular distance of the two parallels.*

Let A be the given point; XY and ZW two parallel straight lines, indefinite in length; and B



a given finite straight line, not less than the perpendicular distance of XY from ZW: It is required to draw through A, a straight line, cutting XY and ZW, so that the part of it, between the two parallels, shall be equal to B.

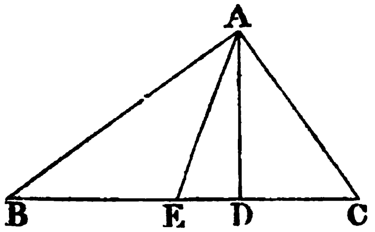
Take any point C in ZW; from C as a centre, at a distance equal to B, describe a circle, cutting XY in D; join C, D; and through A draw (E. xxxiv. 1.) AF parallel to DC, cutting XY and ZW in the points E and F: Then is $EF = B$.

For, (*constr.*) the figure $DCFE$ is a parallelogram; therefore (E. xxxiv. 1.) $EF = DC$; and (*constr.*) $DC = B$; $\therefore EF = B$.

Prop. LIII.

70. Theorem. *If, from the summit of the right angle of a scalene right-angled triangle, two straight lines be drawn, one perpendicular to the hypotenuse, and the other bisecting it, they shall contain an angle equal to the difference of the two acute angles of the triangle.*

Let the $\angle A$, of the $\triangle BAC$ be a right angle; let



AE be drawn to the bisection E, of the hypotenuse BC, and let AD be drawn perpendicular to BC: The $\angle EAD = \angle C - \angle B$.

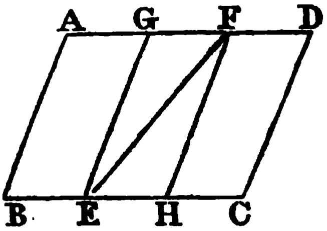
For (Supp. xxix. 1. and *hyp.*) $EA = EB$; therefore the $\angle EAB = \angle EBA$: Again, since (*hyp.*) the two

$\angle$s BAC, CBA, of the $\triangle$ BAC, are equal to the two $\angle$s BDA, ABD, of the $\triangle$ ADB; therefore (Supp. xxvi. 1.) the $\angle BAD = \angle ACB$; but the $\angle EAD = \angle BAD - \angle BAE$; therefore the $\angle EAD = \angle C - \angle B$.

PROP. LIV.

71. PROBLEM. *To bisect a parallelogram by a straight line drawn through a given point in one of its sides.*

Let $ABCD$ be a parallelogram, and E a point



in one of its sides: It is required to bisect the $\square ABCD$, by a straight line drawn through E.

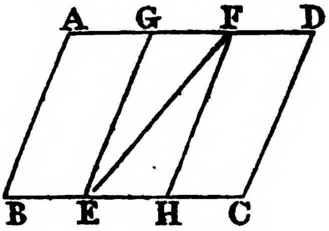
From AD, the side opposite to BC, cut off $DF = BE$ (E. iii. 1.); and join E, F; EF bisects the $\square ABCD$.

For, through E and F draw (E. xxxi. 1.) EG and FH, each parallel to AB or DC; $\therefore$ AE, GH, FC are parallelograms; and since EF is the diameter of the $\square GH$, therefore (E. xxxiv. 1.) the $\triangle EGF = \triangle EHF$; also, because $BE = FD$, and that AD is parallel to BC, therefore (E. xxxvi. 1.) the $\square AE = \square FC$; to these equals add the equal $\triangle$s, EGF, EHF, and it is evident that the trapezium $ABEF$ is equal to the trapezium $FECD$; *i. e.* EF bisects the $\square ABCD$.

PROP. LV.

72. THEOREM. *A trapezium, which has two of its sides parallel, is the half of a rectangle between the same parallels, and having its base equal to the aggregate of the two parallel sides of the trapezium.*

Let *ABEF* be a trapezium, having its side *AF* parallel to the opposite side *BE*: The trapezium



ABEF is equal to the half of a rectangle between *AF* and *BC*, and having its base equal to *AF*+*BE*.

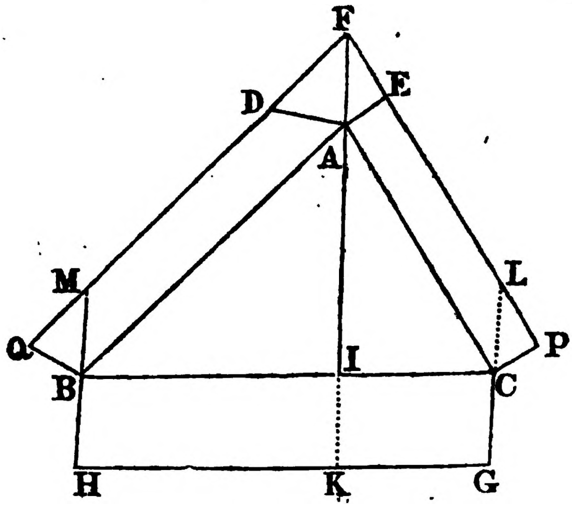
For, produce *BE* to *C*, and make *EC* = *AF*; through *C* draw (E. xxxi. 1.) *CD* parallel to *BA*, and let *CD* meet *AF* produced in *D*; therefore the figure *ABCD* is a parallelogram; therefore (E. xxxiv. 1.) *AD* = *BC*; and (*constr.*) *AF* = *EC*; ∴ *FD* = *BE*: It is manifest, therefore, (from Supp. liv. 1.) that the trapezium *ABEF* is the half of the □ *ABCD*; but (E. xxxv. 1.) the □ *ABCD* = a rectangle upon the same base *BC*, and between the same two parallels; therefore the trapezium *ABEF* = the half of a rectangle on the base *BC*, which (*constr.*) = *BE* + *AF*, and between the two parallels *BE* and *AF*.

PROP. LVI.

73. PROBLEM. *Any two parallelograms having been described on two sides of a given triangle, to*

apply, to the remaining side, a parallelogram, which shall be equal to their aggregate.

Let the ▱ AQ and AP be on the two sides AB, AC, of the given △ ABC: It is required to apply



to the remaining side BC, a parallelogram which shall be equal to the ▱ AP together with the ▱ AQ.

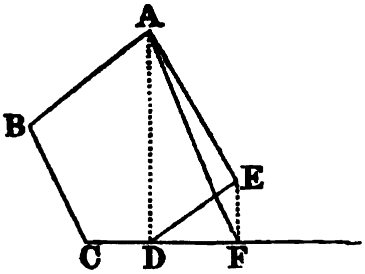
Produce QD and PE until they meet in F; join F, A; through C draw (E. xxxi. 1.) CG parallel to FA, and make, also, $CG = FA$; complete the ▱ $BCGH$: The ▱ $BCGH$ = ▱ AQ + ▱ AP.

For, produce FA, so that it shall meet BC in I, and HG in K; produce, also, GC and HB, until they meet EP and DQ in L and M; therefore the figures $FACL$, $FABM$ are parallelograms; and, since (*constr.*) the ▱ $FACL$, $CGKI$, are upon equal bases FA, CG and between the same parallels; ∴ (E. xxxvi. 1.) the ▱ $FACL$ = ▱ $CGKI$; but (E. xxxv. 1.) the ▱ $FACL$ = ▱ AP; therefore the ▱ AP, = ▱ GI: And in the same manner, it may be proved that the $\square\ AQ = \square\ IH$; therefore the whole $\square\ BCGH = \square\ AP + \square\ AQ$*.

Prop. LVII.

74. Problem. *A plane rectilineal figure of any number of sides being given, to find an equal rectilineal figure, which shall have the number of its sides less, or greater, by one, than that of the given figure.*

First, let *ABCDE* be the given rectilineal figure:



It is required to find an equal rectilineal figure, having the number of its sides less by one, than the number of the sides of *ABCDE*.

Let *A*, *E*, *D* be any three consecutive angles of the given figure *ABCDE*; join *A*, *D*; through *E* draw (E. xxxi. 1.) *EF* parallel to *AD* and meeting *CD*, produced, in *F*; join *A*, *F*: The figure *ABCF*, which has the number of its sides less by one than *ABCDE*, is equal to *ABCDE*.

For, since the two △s *AED*, *AEF*, are upon the same base *AD* and (*constr.*) are between the same parallels, *AD*, *EF*, therefore, (E. xxxvii. 1.) the

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* If the parallelograms *AP* and *AQ* are squares, and the angle at *A* a right angle, it is easy to shew that the parallelogram *BG* will also be a square; and thus the forty-seventh proposition of the first Book of Euclid's Elements will have been demonstrated.

△ *AFD* = △ *AED*; to each of these equals add the figure *ABCD*; and the figure *ABCF* = the figure *ABCDE*.

Secondly, let *ABCF* be a given rectilineal figure; and let it be required to find an equal rectilineal figure, having more sides by one, than *ABCF*.

Take any point, *D*, in any of the sides, as *CF*, of *ABCF*, and join *B*, *D*, or *A*, *D*; *A* and *B* being the angles which are next to the ∠s *F* and *C*, at the extremities of *CF*; then, *A*, *D* having been joined, through *F* draw (E. xxxi. 1.) *FE* parallel to *DA*; and since the ∠ *ADC* is greater (E. xvi. 1.) than the ∠ *AFD*, and equal (E. xxix. 1.) to the ∠ *EFD*; therefore *FE* falls without the given figure: In *FE* take any point *E*, which is not in the continuation of *BA*, the side next to *FA*; and join *E*, *A*, and *E*, *D*: The figure *ABCDE* has more sides, by one, than the given figure *ABCF*; and it may be shewn, as in the preceding case, to be equal to *ABCF*.

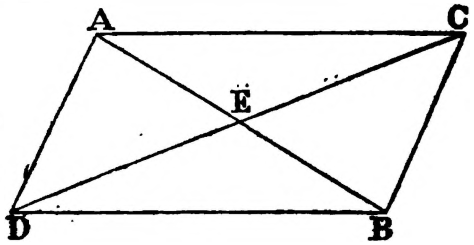
75. Cor. Hence, first, a triangle may be found which shall be equal to any given rectilineal figure: For the number of sides of the given figure being thus diminished, by one, at each step, they will at length be reduced to three, and the triangle which they contain, will be equal to the given figure.

Secondly, it is manifest, that, by the latter part of the preceding problem, a polygon, of any given number of sides, may be found, which shall be equal to a given triangle.

Prop. LVIII.

76. Theorem. *The diameters of any parallelogram divide it into four equal triangles.*

Let *ADBC* be a parallelogram, of which the



diameters *AB*, *CD* cut one another in *E*: The four △s *AED*, *DEB*, *BEC*, *CEA* are equal to one another.

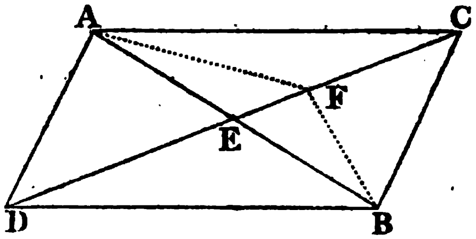
For (*hyp*. and E. xxxiv. 1.) the side *AC* of the △ *AEC*, is equal to the side *DB*, of the △ *DEB*; also (Supp. xlii. 1.) *AE* = *EB*, and *DE* = *EC*; (E. viii. 1. and E. iv. 1.) the △ *AEC* = △ *DEB*. In the same manner, it may be shewn that the △ *AED* = △ *CEB*: And since, the two △s *AED*, *AEC*, stand upon equal bases *DE* and *EC*, therefore (E. xxxviii. 1.) the △ *AED* = △ *AEC*. It is manifest, therefore, that the four △s *AED*, *AEC*, *CEB*, *BED* are equal to one another.

Prop. LIX.

77. **Problem.** *If two triangles have the two adjacent sides of a parallelogram for their bases, and have their common vertex situated in the diameter, or in the diameter produced, they shall be equal to one another.*

Let the two △s *AFC*, *BFC*, have the two adjacent sides *AC*, *BC*, of the ▭ *ADBC*, for their bases, and also have their common vertex situated at any point *F*, in the diameter *DC*, or in *DC*, produced: The △ *AFC* = △ *BFC*.

First, let the point F be in the diameter DC: Join A, B; and let AB cut DC in E.



Then, since (Supp. xlii. 1.) $AE = EB$, therefore (E. xxxviii. 1.) the $\triangle\, AEC = \triangle\, BEC$, and the $\triangle\, AEF = \triangle\, BEF$; $\therefore$ the $\triangle$s AFC, BFC, which are the differences of these equals, are equal to one another.

And the proposition may, in the same manner, be shewn to be true, when the common vertex of the two triangles, which have AC and BC for their bases, is in DC produced.

PROP. LX.

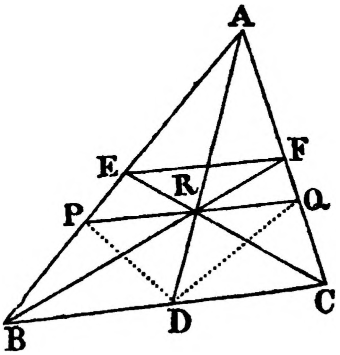
78. THEOREM. *Of all triangles, which are between the same parallels, that which stands on the greatest base is the greatest.* .

For it is manifest, that the triangle which has the greater base will exceed the triangle which is formed by joining its vertex and the extremity of a segment of its base made equal to the base of the other triangle: But the triangle so formed is equal (E. xxviii. 1.) to the other given triangle; therefore the triangle which has the greater base is greater than that other triangle.

PROP. LXI.

79. THEOREM. *The straight line, joining the vertex and the bisection of the base of any triangle, bisects every other straight line that is parallel to the base and is terminated by the two remaining sides of the triangle, or those sides produced.*

Let *PQ* be any straight line, either within or without the △ *ABC*, parallel to the base *BC*, and let *AD*, joining the vertex *A* and the bisection *D* of *BC*, cut *PQ* in *R*: *PQ* is bisected by *AD* in *R*.



First, let *PQ* be within the △ *ABC*; and if *PR* be not equal to *RQ*, one of them is the greater: Let *PR* > *RQ*; and join *D*, *P*, and *D*, *Q*.

Then since (*hyp.*) the base *BD*, of the △ *BAD*, is equal to the base *DC*, of the △ *CAD*, therefore (E. xxxviii. 1.) the △ *BAD* = △ *CAD*; also, because *BD* = *DC*, and that (*hyp.*) *PQ* is parallel to *BC*; therefore (E. xxxviii. 1.) the △ *BPD* = △ *CQD*; if, therefore, the two latter equal triangles be taken from the equal △s *BAD*, *CAD*, there remains the △ *APD* = △ *AQD*: But, since *PR* > *RQ*, the △ *APR* > △ *AQR*, and the △ *DPR* > △ *DQR*; therefore, the whole △ *APD* > △ *AQD*; but it has been shewn that the △ APD = △ AQD; and it is, also, greater; which is absurd: therefore, neither of the two lines PR, RQ, can be greater than the other; therefore, $PR = RQ$.

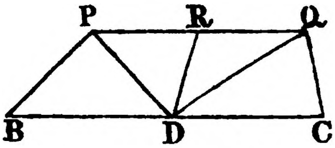
In a similar manner the proposition may be proved, when PQ is without the △ ABC.

80. Cor. Hence, it is easily shewn, *ex absurdo*, that the straight line joining the bisections of any two straight lines, that are parallel to the base, and terminated by the sides of a triangle, passes through the vertex of the triangle.

Prop. LXII.

81. Theorem. *If two opposite sides of a trapezium be parallel to one another, the straight line, joining their bisections, bisects the trapezium.*

For, let $PBCQ$ be a trapezium having the side



PQ parallel to BC, and let $\overline{RD}$ join the bisections, R and D, of the opposite sides PQ and BC : RD bisects the trapezium $PBCQ$.

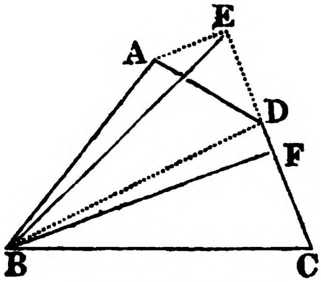
For, join P, D, and Q, D: Then since (*hyp.*) PQ is parallel to BC, and that the base BD of the △ BPD, is equal to the base DC, of the △ DQC, therefore (E. xxxviii. 1.) the △ BPD = △ DQC; and, in the same manner, it may be shewn that the △ PDR = △ DRQ; therefore, △ BPD + △ PDR = △ DQC + △ DRQ; *i. e.* the figure $BPRD$ = $CQRD$; ∴ RD bisects the trapezium $PBCQ$.

F

PROP. LXIII.

82. PROBLEM. *To bisect a given trapezium by a straight line drawn from any of its angles.*

Let *ABCD* be a trapezium: It is required to draw a straight line from any of the angles, as *B*, which shall bisect the trapezium *ABCD*.



Join *B*, *D*; through *A* draw (E. xxxi. 1.) *AE* parallel to *BD*, and let *CD*, produced, meet *AE* in *E*; bisect (E. x. 1.) *EC* in *F*; and join *B*, *F*; *BF* bisects the trapezium *ABCD*.

For join *B*, *E*; and since the two △s *BAD*, *BED* are on the same base *BD*, and between the same parallels, therefore (E. xxxvii. 1.) the △ *BAD* = △ *BED*; to each of these equals add the △ *BDF*; ∴ △ *BAD* + △ *BDF* = △ *BED* + △ *BDF*; *i. e.* the trapezium *BADF* = △ *BEF*; but since (*constr.*) *EF* = *FC*, therefore (E. xxxviii. 1.) the △ *BEF* = △*BFC*: therefore the trapezium *BADF* = △*BFC*; *i. e. BF* bisects the given trapezium *ABCD*.

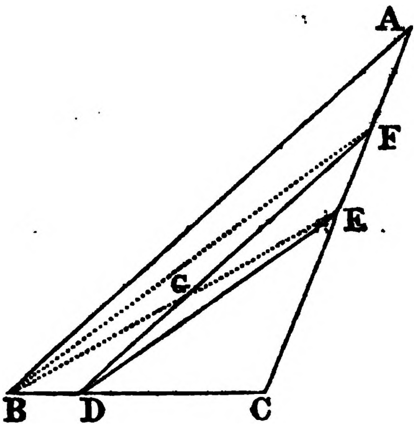
PROP. LXIV.

83. PROBLEM. *To bisect a given triangle by a straight line drawn through a given point in any one of its sides.*

Let *ABC* be the given triangle, and let *D* be a point in one of its sides BC: It is required to draw through D a straight line which shall bisect the triangle.

Bisect (E. x. 1.) AC in E; join D, E; through B draw (E. xxxi. 1.) BF parallel to DE, meeting AC in F; join D, F: $\overline{DF}$ bisects the △ ABC.

For join B, E and let BE cut DF in G: Then since the △s DFE, EBD are upon the same base DE and (*constr.*) between the same parallels, therefore (E. xxxvii. 1.) the △ DFE = △ EBD; take away the common part DGE, and there remains the △ BGD = △ EGF; to each of these equals add the trapezium $ABGF$, and it is manifest that the trapezium $ABDF$ = △ ABE; but since (*constr.*) $AE = EC$, therefore (E. xxxviii. 1.) the △ ABE = △ EBC; therefore the trapezium $ABDF$ is equal to the half of the given △ ABC; *i. e.* DF bisects the △ ABC.



Prop. LXV.

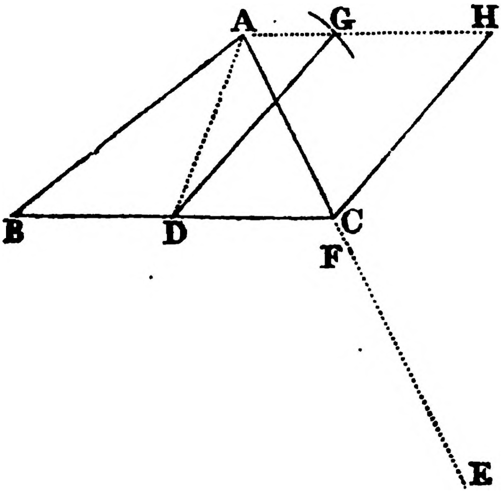
84. Problem. *Equal triangles, which have their bases in the same straight line and which are between the same parallels, stand upon equal bases.*

For if not, let one of the bases be greater than the other; therefore (Supp. lx. 1.) the triangle, of which it is the base, is greater than the other, which is contrary to the supposition: therefore, neither of the bases can be greater than the other; *i. e.* the bases are equal to one another.

Prop. LXVI.

85. Problem. *To describe a parallelogram, the surface and perimeter of which shall be respectively equal to the surface and perimeter of a given triangle.*

Let ABC be the given triangle: It is required to



describe a parallelogram, which shall be equal to the $\triangle ABC$, and which shall, also, have its perimeter equal to the perimeter of ABC.

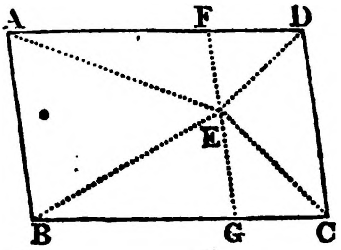
Bisect (E. x. 1.) BC in D; produce AC to E, and make $CE=AB$; bisect AE in F; through A draw (E. xxxi. 1.) AH parallel to BC; from D as a centre, at a distance equal to AF describe a circle, cutting AH in G; join D, G, and through C draw $\overline{CH}$ parallel to $\overline{DG}$: Then is the ▱ $DCHG = \triangle ABC$, and the perimeter of ▱ $DCHG$ is, also, equal to the perimeter of $\triangle ABC$.

For join A, D; and since $BD = DC$, therefore (E. xxxviii. 1.) the $\triangle ABD = \triangle ACD$, so that the whole $\triangle ABC$ is the double of the $\triangle ADC$: Again, since the ▱ $DCHG$ and the $\triangle ADC$ are on the same base DC, and between the same parallels, therefore (E. xli. 1.) the ▱ $DCHG$ is the double of the $\triangle ADC$; as is, also, the $\triangle ABC$: therefore the ▱ $DCHG = \triangle ABC$: And because (*constr.*) DG is equal to the half of $BA + AC$, and that (E. xxxiv. 1.) $CH = DG$, $\therefore DG + CH = BA + AC$; also (E. xxxiv. 1.) $GH = DC = DB$; $\therefore DC + GH = BD + DC = BC$; $\therefore DG + GH + HC + CD = BA + AC + CB$.

Prop. LXVII.

86. **Theorem.** *The two triangles formed by drawing straight lines, from any point within a parallelogram, to the extremities of either pair of opposite sides, are, together, half of the parallelogram.*

Let E be any point in the ▱ $ABCD$, and let



E, A, and E, B, and E, C and E, D be joined:

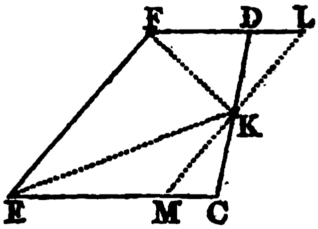
The two △s *AEB*, *DEC* are, together, half of the ▭*ABCD*.

For, through *E* draw (E. xxxi. 1.) $\overline{FEG}$ parallel to $\overline{AB}$ or $\overline{DC}$: and since *AG*, and *GD* are parallelograms, therefore (E. xli. 1.) the △*AEB* is the half of the ▭*AG*, and the △*DEC* is the half of the ▭*GD*; therefore the △*AEB*+△*DEC* is the half of the ▭*AG*+the half of the ▭*GD*, or the half of the whole ▭*ABCD*.

Prop. LXVIII.

87. Theorem. ***If two sides of a trapezium be parallel, the triangle contained by either of the other sides, and the two straight lines drawn from its extremities to the bisection of the opposite side, is the half of the trapezium.***

Let the two sides *FD*, *EC*, of the trapezium *FECD* be parallel; let *K* be the bisection of either



of the two remaining sides, as *DC*; and let *K*, *E* and *K*, *F* be joined: The △*FKE* is the half of the trapezium *FECD*.

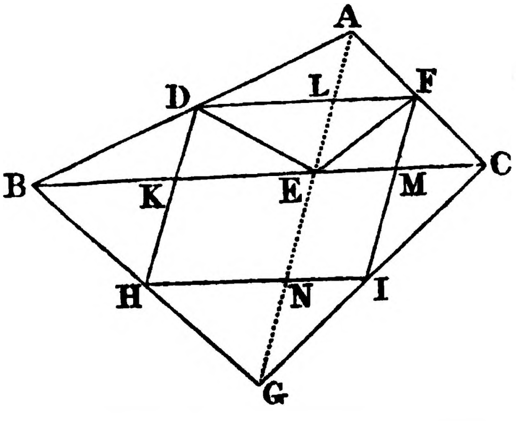
For, through *K* draw (E. xxxi. 1.) $\overline{LKM}$ parallel to *FE*, and let *LKM* meet *BC* in *M* and *FD*, produced, in *L*. And since *FL* (*constr.*) is parallel to *EC*, and *LM* meets them, therefore the ∠*DLK*=

$\angle KMC$; also (E. xv. 1.) the $\angle DKL = \angle CKM$, and (*hyp.*) the side DK of the $\triangle DKL$, = the side CK of the $\triangle CKM$; therefore (E. xxvi. 1. and E. iv. 1.) the $\triangle DKL = \triangle CKM$; but if to the rectilineal figure $FEMKD$ there be added the $\triangle CKM$, there results the trapezium $FECD$; and if to the same figure there be added the $\triangle DKL$, there results the ▭$FEML$; therefore these results are equal; but (E. xli. 1.) the $\triangle FKE$ is the half of the ▭$FEML$; therefore, the $\triangle FKE$ is the half, also, of the trapezium $FECD$.

PROP. LXIX.

88. THEOREM. *The triangle contained by the straight lines joining the points of the bisection of the three sides of a given triangle, is one-fourth part of the given triangle, and is equiangular with it.*

Let D, E, F, be the bisections of the sides AB,



BC, CA, respectively, of the given $\triangle ABC$; and let D, E, and E, F, and F, D, be joined: The $\triangle DEF$

is one fourth part of the $\triangle ABC$, and is equiangular with it.

For, join A, E; and, since (*hyp.*) $BE=EC$, $CF=FA$, and $AD=DB$, therefore (E. xxxviii. 1.) $\triangle AEB=\triangle AEC$; and the $\triangle AEB$ is the double of the $\triangle BDE$, and the $\triangle AEC$ is the double of the $\triangle CFE$; therefore the $\triangle BDE=\triangle CFE$; *i.e.* each of them is a fourth part of the $\triangle ABC$; also they are upon equal bases BE and EC; therefore (E. xl. 1.) DF is parallel to BC; and, in the same manner, it may be shewn that DE is parallel to AC, and FE parallel to AB; therefore, the figures $FCED$, $DBEF$, are parallelograms; therefore (E. xxxiv. 1.) the $\triangle DEF=\triangle DBE$, which has been proved to be a fourth part of the $\triangle ABC$; also, the $\angle DFE$, of the $\square BF$, $=$opposite $\angle B$, and the $\angle FDE$, of the $\square DC$, $=$opposite $\angle C$; therefore (E. xxxii. 1.) the $\angle DEF$, of the $\triangle DFE=$the $\angle BAC$, of the $\triangle ABC$; and the two $\triangle$s ABC, DEF, are therefore equiangular.

89. Cor. 1. The straight line joining the bisections of any two sides of a triangle, is parallel to the remaining side.

90. Cor. 2. If the four sides of any given quadrilateral rectilineal figure be bisected, the figure contained by the straight lines joining the several points of the bisection, shall be a parallelogram, which is the half of the given figure; also the four sides of this parallelogram shall be, together, equal to the two diagonals of the given figure.

Let DH, HI, IF, FD be the straight lines joining the several bisections D, H, I, and F, of the sides AB, BG, GC, and CA, of the quadrilateral figure

$ABGC$: The figure $DHIF$ is a parallelogram; it is the half of the given figure $ABGC$; and its four sides are, together, equal to the two diagonals AG, BC, of the figure $ABGC$.

First, since, D, H, F, I, are the bisections of the sides of the $\triangle$s ABG, GCA, BAC, CGB, therefore (Supp. lxix. 1. *Cor.*) DH and FI are parallel to AG, and DF and HI are parallel to BC; therefore (E. xxx. 1.) $DHIF$ is a parallelogram: And, because DF is parallel to BC, and AB meets them, therefore (E. xxix. 1.) the $\angle ADL = \angle DBK$; again, because DH is parallel to $\overline{AG}$, and $\overline{AB}$ meets them, the $\angle DAL = \angle BDK$; and (*hyp.*) the side AD, of the $\triangle ADL$, $=$ the side DB of the $\triangle DBK$; therefore (E. xxvi. 1.) $DL = BK$, $LA = KD$, and (E. iv. 1.) the $\triangle ADL = \triangle DBK$; but $DKEL$ being a parallelogram, $DL = KE$, and $KD = EL$ (E. xxxiv. 1.); $\therefore BK = KE$, and $EL = LA$: If, therefore, D, E be joined, the $\triangle DLE = \triangle DLA$ (E. xxxviii. 1.) and the $\triangle DKE = \triangle DKB$; so that the $\square KL =$ the half of the $\triangle AEB$, $DK + FM = AE$, and $DL + HN = BE$. In the same manner it may be proved, that the $\square LM =$ the half of the $\triangle AEC$, that the $\square MN =$ the half of the $\triangle CEG$, that the $\square NK =$ the half of the $\triangle BEG$, that $LF + NI = EC$, and that $MI + KH = EG$: therefore, the $\square DHIF$ is the half the given figure $ABGC$, and its four sides are, together, equal to the two diagonals AG, and BC.

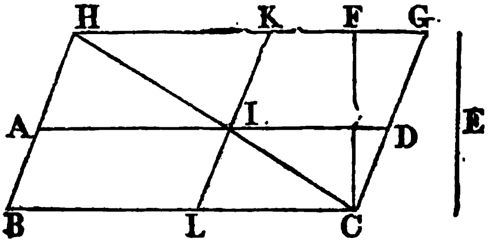
91. Cor. 3. It is manifest that the straight lines which join the opposite points of bisection of the sides of any trapezium, bisect each other.

For, if D, I, and F, H, be the bisections of opposite sides of the given quadrilateral figure $ABGC$, it is manifest, from the preceding corollary, that the straight lines *DI*, *FH* which join them, will be the diameters of the ▱ *DHIF*; and therefore (Supp. xlii. 1.) they bisect one another.

Prop. LXX.

92. Problem. *To describe a parallelogram, which shall be of a given altitude, and equiangular with, and also equal to, a given parallelogram.*

Let *ABCD* be a parallelogram, and *E* a straight line: It is required to describe a parallelogram which shall be equal to the ▱ *ABCD*, and also equiangular with it; and which shall have its altitude equal to the given line *E*.



From the point *C* draw (E. xi. 1.) *CF* perpendicular to *BC*, and make $CF = E$; through *F* draw (E. xxxi. 1.) *HG* parallel to *BC*; produce *BA* and *CD* to meet *HG*, in *H* and *G*; join *H*, *C*, and let *HC* cut *AD* in *I*; through *I* draw (E. xxxi. 1.) *KIL* parallel to *HB* or *GC*: The ▱ *KLCG*, which (*constr.* and E. xxix. and xxxiv. 1.) is equiangular with the ▱ *ABCD*, and has its altitude equal to *E*, is also equal to the ▱ *ABCD*.

For, since *BI* and *IG* are complements about the diameter *HC* of the ▱ *HBCG*, they are (E. xliii. 1.)

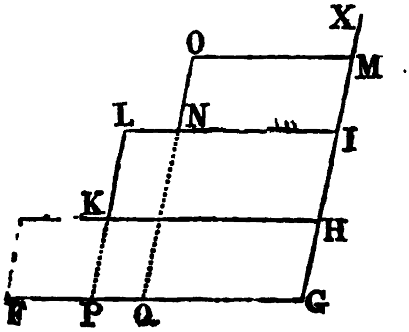
equal to one another; to each of these equals add the ▭LD; and it is plain that the ▭$KLCG=$ ▭$ABCD$.

93. Cor. Hence, a rectangle may very readily be found, which shall be equal to a given square, and shall have one of its sides equal to a given straight line.

Prop. LXXI.

94. Theorem. *If there be any number of rectilineal figures, of which the first is greater than the second, the second than the third, and so on, the first of them shall be equal to the last together with the aggregate of all the differences of the figures.*

First let there be three such given rectilineal figures. Make (E. xlv. 1.) the ▭FH equal to the greatest of the given figures, having its $\angle FGH$ of any given magnitude; produce GH to X; from HX cut off (E. iii. 1.) $HI=GH$; find (Supp. lvii. 1. Cor.) a triangle equal to the next greatest of the given figures, and apply (E. xliv. 1.) to HI a parallelogram equal to that triangle, having its $\angle IHK=\angle HGF$: Again, from IX cut off $IM=GH$ or HI,



and, in like manner, to *IM* apply a □ *IO*, equal to the least of the given figures, and having its ∠*MIN* = ∠*HGF*. Produce *LK* and *ON* to meet *FG* in *P* and *Q*; and let *OQ* meet *KH* in *R*.

Then, (E. xxxvi. 1. E. xxxiv. 1. and *constr.*) the □

$$FH = \square QH + \square PR + \square FK$$

i. e. the $\square FH = \square NM + \square PR + \square FK$.

But the □*PR* is the difference of the □s *PH* and *QH* or (E. xxxvi. 1.) of *KI* and *NM*; and the □*FK* is the difference of the □s *FH* and *PH*, or of *FH* and *KI*: Whence it is manifest that the proposition is true, when three rectilineal figures are taken: And it may, in the same manner, be proved to be true, when more than three are taken.

PROP. LXXII.

95. PROBLEM. *To find a rectangle, which shall have one of its sides equal to a given finite straight line, and which shall be equal to the excess of the greater of two given rectilineal figures above the less.*

To the given finite straight line, and on the same side of it, apply (E. xlv. 1. *Cor.*) two rectangles, the one equal to the greater and the other to the less, of the given rectilineal figures: And it is manifest that the rectangle which is the difference of the two rectangles so described, will have one of its sides equal to the given straight line, and will be equal to the excess of the greater of the two given figures above the less.

Prop. LXXIII.

96. **Theorem.** *If two right-angled triangles have two sides of the one equal to two sides of the other, each to each, the triangles shall be equal, and similar to each other.*

If the two sides about the right angle of the one triangle, be equal to the two sides about the right angle of the other, each to each, it follows, (from E. iv. 1.) that the triangles are equal and similar.

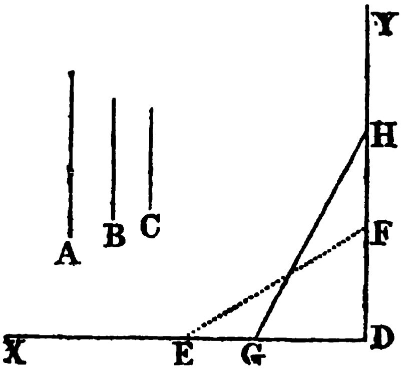
But, let now, the hypotenuses of the right angles, in the two triangles, be equal, and also let one other side, of the one triangle, be equal to another side of the other; therefore (E. xlvii. 1.) the squares of the two remaining sides of the one, will be equal to the squares, taken together, of the two remaining sides of the other triangle; from these equals take away the equal squares of the two other sides, which, by the hypothesis, are equal, and there remains the square of the third side, of the one, equal to the square of the third side, of the other triangle; therefore the third side of the one is equal to the third side of the other; therefore (E. iv. 1.) the two triangles are equiangular, and are, also, equal to one another.

Prop. LXXIV.

97. **Problem.** *To find a square which shall be equal to any number of given squares.*

First, let there be three given squares, and let their sides be equal to the three straight lines *A, B* and *C*.

Take any straight line DX, indefinite towards X; from D draw (E. xi. 1.) $\overline{DY}$ perpendicular to $\overline{DX}$,



and produce DY indefinitely towards Y: From DX cut off (E. iii. 1.) $DE = A$, and from DY cut off $DF = B$; and join E, F: Again, from DY cut off $DH = EF$, and from DX cut off $DG = C$, and join G, H: The square described (E. xlvi. 1.) upon GH shall be equal to the three given squares to the sides of which A, B and C are respectively equal.

For (E. xlvii. 1. and *constr.*) $\overline{EF}^2 = \overline{ED}^2 + \overline{DF}^2$;

i. e. (*constr.*) $\overline{DH}^2 = A^2 + B^2$;

∴ (*constr.*) $\overline{DH}^2 + \overline{DG}^2 = A^2 + B^2 + C^2$

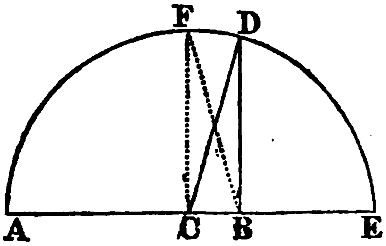
i. e. (E. xlvii. 1.) $\overline{GH}^2 = A^2 + B^2 + C^2$.

And in the same manner, it is evident, a square may be found, which shall be equal to the aggregate of any number of given squares.

PROP. LXXV.

98. PROBLEM. *Two unequal squares being given, to find a third square, which shall be equal to the excess of the greater of them above the less.*

Let AC and CB placed in the same straight line,



be the sides of the two given squares, of which the square of AC is the greater: From the centre C, at the distance CA, describe the circle ADE, meeting AB, produced, in E; from B draw (E. xi. 1.) $\overline{BD}$ perpendicular to AB, and let BD meet the circumference in D: The square of BD is equal to the excess of the square of AC above the square of BC.

For join D, C: And since (*constr.*) the $\angle B$ is a right angle; therefore (E. xlvii. 1.) $\overline{CD}^2=\overline{CB}^2+\overline{BD}^2$

i. e. (E. Def. 15. 1.) $\overline{AC}^2=CB^2+BD^2$.

Whence it is manifest, that the square of BD is equal to the excess of the square of AC above the square of CB.

PROP. LXXVI.

99. **THEOREM.** *If the side of a square be equal to the diameter of another square, the former square shall be the double of the latter.*

For (E. Def. 30. 1. and E. xlvii. 1.) the square of the diameter of a square is equal to the squares of its two sides; *i. e.* to the double of the square itself: therefore the square of any straight line which is equal to the diameter of a square, is the double of that square.

PROP. LXXVII.

100. THEOREM. *In any right-angled triangle, the square which is described on the side subtending the right angle, as a diameter, is equal to the squares described upon the other two sides, as diameters.*

For, (Supp. lxxvi. 1.) the squares described on the hypotenuse, and on the two sides of a triangle as diameters, are, respectively, the halves of the squares of those lines: But since (*hyp.*) the triangle is right angled, therefore (E. xlvii. 1.) the square of the hypotenuse is equal to the squares of the two sides; therefore the square described on the hypotenuse as a diameter, is equal to the squares described on the other two sides as diameters.

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A
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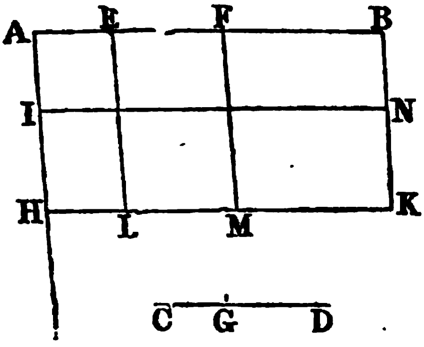
PART I.

BOOK II.

Prop. I.

1. **Theorem.** *If two given straight lines be divided, each into any number of parts, the rectangle contained by the two straight lines, is equal to the rectangles contained by the several parts of the one and the several parts of the other.*

Let the given straight line AB be divided into



any parts in the points *E, F,* and let the given straight line *CD* be divided first into two parts in the point *G*: The rectangle contained by $\overline{AB}$ and $\overline{CD}$ is equal to rectangles contained by *AE* and *CG,* by *EF* and *CG,* by *FB* and *CG,* by *AE* and *GD,* by *EF* and *GD,* and by *FB* and *GD,* taken together.

From the point *A* draw (E. xi. 1.) $\overline{AX}$ perpendicular to *AB*; from *AX* cut off (E. iii. 1.) *AI*=*CG,* and from *IX* cut off *IH*=*GD,* so that *AH*=*CD*; through *I* and *H* draw (E. xxxi. 1.) $\overline{IN}$ and $\overline{HK}$ parallel to *AB,* and through *B, F, E,* draw $\overline{BK}$, $\overline{FM}$, $\overline{EL}$, parallel to *AH*: Then (E. i. 2.) the rectangle *AN* is equal to the rectangles contained by *AE* and *CG,* by *EF* and *CG,* and by *FB* and *CG*; also the rectangle *IK* is equal to the rectangles contained by *HL* and *GD,* by *LM* and *GD,* and by *MK* and *GD*; but (E. xxxiv. 1.) *HL*=*AE*; *LM*=*EF*; and *MK*=*FB*; therefore the rectangle *IK* is equal to the rectangles contained by *AE* and *GD,* by *EF* and *GD,* and by *FB* and *GD*; but the two rectangles *AN* and *IK* make up the rectangle *AK,* which is contained by *AB* and *AH* or *CD*; therefore the rectangle contained by *AB* and *CD* is equal to the rectangles contained by *AE* and *CG,* by *EF* and *CG,* by *FB* and *CG,* by *AE* and *GD,* by *EF* and *GD,* and by *FB* and *GD,* taken together.

And, in the same manner, the proposition may be proved to be true, when the given straight line *CD* is divided into more than two parts.

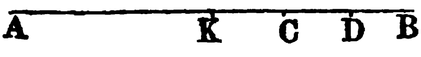
2. Cor. If the parts *EF, FB,* &c., into which

AB is divided, and the parts *CG*, *GD*, &c., into which *CD* is divided, be each of them equal to *AE*, it is manifest that the rectangle contained by *AB* and *CD* is equal to the square of *AE* taken as often as is indicated by the product of the number of equal parts in *AB*, multiplied by the number of equal parts in *CD*.

Prop. II.

3. **Theorem.** *If a straight line be divided into two unequal parts, in two different points, the rectangle contained by the two parts, which are the greatest and the least, is less than the rectangle contained by the other two parts; the squares of the two former parts, together, are greater than the squares of the two latter, taken together; and the difference between the squares of the former and the squares of the latter, is the double of the difference between the two rectangles.*

Let the given straight line *AB* be divided into



two unequal parts, in the point *C*, and also in the point *D*: Then $\overline{AD}\times\overline{DB}<\overline{AC}\times\overline{CB}$; but $AD^2+DB^2>AC^2+\overline{BC}^2$; and the excess of AD^2+DB^2 above AC^2+CB^2 is the double of the excess of $AC\times CB$ above $AD\times DB$.

For, bisect (E. x. 1.) *AB* in *K*: Therefore,

$$\left.\begin{array}{r}AC\times CB+\overline{CK}^2=\overline{AK}^2\\ \text{and } AD\times DB+\overline{DK}^2=AK^2\end{array}\right\}\text{(E. v. 2.)}$$

But $CK^2<DK^2$; $\therefore AD\times DB<AC\times CB$.

Again, because

$$\left.\begin{aligned} AD^2 + DB^2 + 2AD \times DB &= AB^2 \\ AC^2 + CB^2 + 2AC \times CB &= AB^2 \end{aligned}\right\} \text{(E. iv. 2.)}$$

and that, as has been shewn $AD \times DB < AC \times CB$.

$$\therefore AD^2 + DB^2 > AC^2 + CB^2.$$

Lastly, since

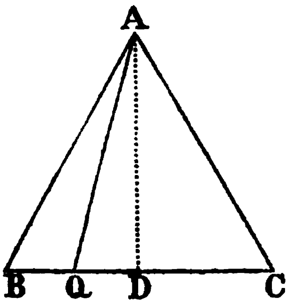
$$AD^2 + DB^2 + 2AD \times DB$$
$$= AC^2 + CB^2 + 2AC \times CB,$$

it is manifest, if from these equals there be taken $AC^2 + CB^2 + 2AD \times DB$, that the excess of $AD^2 + DB^2$ above $AC^2 + CB^2$ is the double of the excess of $AC \times CB$ above $AD \times DB$.

PROP. III.

4. THEOREM. *In any isosceles triangle, if a straight line be drawn from the vertex to any point in the base, the square upon this line, together with the rectangle contained by the segments of the base, is equal to the square upon either of the equal sides.*

Let ABC be an isosceles triangle, and let $\overline{AQ}$, be



drawn from its vertex A, to any point Q, in BC its base: $AQ^2 + \overline{BQ} \times \overline{QC} = \overline{AB}^2$.

For bisect (E. x. 1.) BC in D, and join A, D.

$\therefore \overline{QD}^2 + BQ \times QC = \overline{BD}^2$ (*constr.* and E. v. 2.)

To each of these equals add DA^2;

$$\therefore \overline{AD}^2+QD^2+BQ\times QC=AD^2+\overline{DB}^2:$$

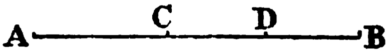
But (*constr.*) $BD=DC$, and DA is common to the $\triangle$s ADB, ADC, and (*hyp.*) $AB=AC$; therefore the $\angle\ ADB=\angle\ ADC$; and therefore each of these angles is a right angle; therefore (E. xlvii. 1.) $AD^2+\overline{DQ}^2=AQ^2$, and $AD^2+DB^2=AB^2$;

$$AQ^2+BQ\times QC=AB^2.$$

PROP. IV.

5. **THEOREM.** *The rectangle contained by the aggregate and the difference of two unequal straight lines is equal to the difference of their squares.*

Let AC and CB be two unequal straight lines,



of which CB is the greater; and let them be placed in the same straight line AB; so that AB is the aggregate of AC, CB, and if (E. iii. 1.) CD be cut off from CB equal AC, DB is the difference between AC and CB. Then since (*constr.* and E. vi. 2.)

$$\overline{AB}\times\overline{DB}+\overline{AC}^2=\overline{CB}^2,$$

it is manifest, if from these equals AC^2 be taken,

$$\text{that } AB\times DB=CB^2-AC^2;$$

i. e. the rectangle contained by the aggregate AB, of AC and CB, and their difference DB, is equal to the difference of their squares.

6. **COR.** If there be three straight lines, the difference between the first and second of which is equal to the difference between the second and third, the rectangle contained by the first and third, is less

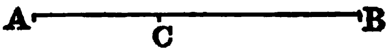
than the square of the second, by the square of the common difference between the lines.

For, let AB, CB, and DB be the three straight lines, having AC the difference of AB and CB, equal to CD, the difference of CB and DB: Then, since it has been shewn that $AB \times DB = CB^2 - AC^2$, it is manifest that $AB \times DB$ is less than CB^2 by AC^2.

PROP. V.

7. THEOREM. *The square of the excess of the greater of two given straight lines above the less, is less than the squares of the two lines, by twice the rectangle contained by them.*

For let AB and CB be two straight lines, of

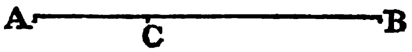


which AB is the greater: Then is AC the excess of AB above CB; and since (E. vii. 2.) $\overline{AC}^2 + 2\overline{AB} \times \overline{BC} = \overline{AB}^2 + \overline{CB}^2$, it is manifest that AC^2 is less than $AB^2 + CB^2$ by $2AB \times BC$.

PROP. VI.

8. THEOREM. *The squares of any two unequal straight lines are, together, greater than twice the rectangle contained by those lines.*

For let AB and CB be two straight lines of



which AB is the greater: Then since (E. vii. 2.)

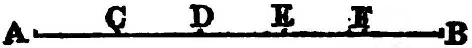
$$\overline{AB}^2 + \overline{CB}^2 = 2AB \times \overline{BC} + \overline{AC}^2$$

it is manifest that $AB^2 + CB^2 > 2AB \times BC$.

PROP. VII.

9. THEOREM. *If a straight line be divided into five equal parts, the square of the whole line is equal to the square of the straight line, which is made up of four of those parts, together with the square of the straight line which is made up of three of those parts.*

Let the straight line AB be divided into five equal



parts by the points C, D, E, F: Then, $\overline{AF}^2+\overline{AE}^2=\overline{AB}^2$.

For since (*hyp.*) $EF=FB$;

$$\therefore 4\overline{FE}\times AF+AE^2=AB^2 \text{ (E. viii. 2.)}$$

But, since $AC=CD=DE=EF$, $4FE=AF$;

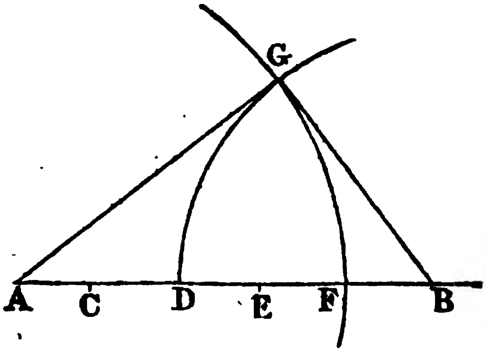
$$\therefore 4FE\times AF=AF^2;$$

$$\therefore AF^2+AE^2=AB^2.$$

PROP. VIII.

10. PROBLEM. *Upon a given straight line, as an hypotenuse, to describe a right-angled triangle, such that the hypotenuse, together with the less of the two remaining sides, shall be the double of the greater of those sides.*

Let AB be the given straight line: Upon AB, as



an hypotenuse, it is required to describe a right-angled triangle, having the less of its two remaining sides, together with *AB*, the double of the third side*.

Divide (Supp. xlix. 1.) *AB* into five equal parts in the points *C, D, E, F*; from *A* as a centre, at the distance *AF*, describe the circle *FG*, and from *B* as a centre at the distance *BD*, describe the circle *DG* cutting *FG* in *G*; join *A, G* and *B, G*: The △ *AGB* is right-angled at *G*, and $AB+BG$ is the double of *AG*.

For (*constr*. and Supp. vii. 2.)

$$\overline{AB}^2=\overline{AF}^2+\overline{AE}^2$$
$$=AF^2+\overline{BD}^2 \text{ (constr.)}$$
$$=\overline{AG}^2+\overline{BG}^2 \text{ (constr. and E. def. 15. 1.)}$$

Wherefore (E. xlviii. 1.) the △ *AGB* is right-angled at *G*: And since (*constr.*) *AB*, and *BG*, together contain eight of such equal parts as *AG* contains four, it is manifest that $AB+BG$ is the double of *AG*.

PROP. IX.

11. **THEOREM.** *In any triangle, the squares of the two sides are, together, the double of the squares of half the base, and of the straight line joining its bisection and the opposite angle.*

Let *ABC* be any triangle, of which *BC* is the base, and *AE* the straight line joining the vertex *A*, and the bisection *E* of the base: Then,

$$\overline{AB}^2+\overline{AC}^2=2\overline{AE}^2+2\overline{EB}^2.$$

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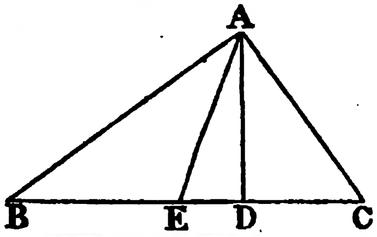
* That is, "Upon a given straight line, as an hypotenuse, to describe a right-angled triangle, the sides of which shall be arithmetic proportionals."

For from A draw (E. xii. 1.) AD perpendicular to BC, and first let AD fall within the base BC.



Then, $\overline{BD}^2+\overline{DC}^2=2\overline{DE}^2+2EB^2$. (E. ix. 2.)

Add to these equals $2\overline{AD}^2$.

$\therefore\ BD^2+DC^2+2AD^2=2\overline{AD}^2+2DE^2+2EB^2$.

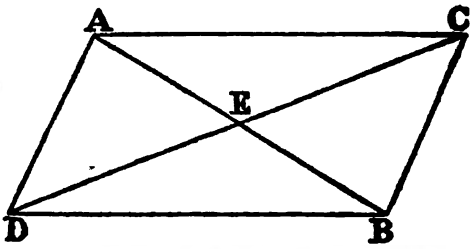
i. e. $AB^2+AC^2=2AE^2+2EB^2$. (E. xlviii. 1.)

And, if the perpendicular AD fall without the base BC, the proposition may, in like manner, be deduced from E. xlvii. 1, and E. x. 2.

Prop. X.

12. **Theorem.** *The squares of the sides of any parallelogram are, together, equal to the squares of its diameters taken together.*

Let $ACBD$ be a parallelogram, of which AB and



CD are the diameters: $\overline{AC}^2+\overline{CB}^2+\overline{BD}^2+\overline{DA}^2=\overline{AB}^2+\overline{CD}^2$.

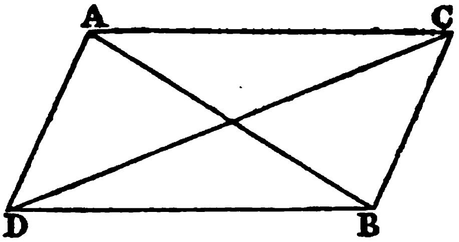
For (Supp. xlii. 1.) AB and CD bisect one another in E:

$\therefore AC^2 + CB^2 + BD^2 + DA^2 = 2\overline{AE}^2 + 2\overline{DE}^2 + 2\overline{BE}^2 + 2DE^2$ (Supp. ix. 2.) $= AB^2 + CD^2$ (E. iv. 2, and Supp. xlii. 1.)

PROP. XI.

13. THEOREM. *If either diameter of a parallelogram be equal to one of the sides about the opposite angle of the figure, its square shall be less than the square of the other diameter, by twice the square of the other side about that opposite angle.*

Let the diameter AB of the ▭ $ACBD$ be equal



to one of the sides, as AC, about the opposite $\angle ACB$; and let CD be the other diameter: Then $\overline{CD}^2 = \overline{AB}^2 + 2\overline{CB}^2$.

For, $CD^2 + AB^2 = 2\overline{AC}^2 + 2CB^2$ (Supp. x. 2, and E. xxxiv. 1.)

From these equals take AB^2 which (*hyp.*) is equal to AC^2; and there remains,

$$CD^2 = AC^2 + 2CB^2:$$

$$i.\ e.\ CD^2 = AB^2 + 2CB^2:$$

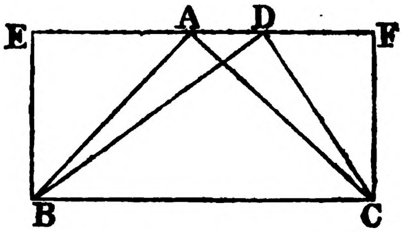
Wherefore AB^2 is less than CD^2 by $2CB^2$.

PROP. XII.

14. THEOREM. *If two sides of a trapezium be parallel to each other, the squares of its diagonals*

are, together, equal to the aggregate of the squares of its two sides, which are not parallel, and of twice the rectangle of its parallel sides.

Let $ABCD$ be a trapezium, having the side AD



parallel to the side BC, and let $\overline{AC}$ and $\overline{BD}$ be its diameters: Then,

$$AC^2+BD^2=\overline{AB}^2+\overline{DC}^2+2\overline{AD}\times\overline{BC}.$$

From B and C, the extremities of BC, the greater of the two parallel sides, draw (E. xii. 1.) BE and CF, each perpendicular to AD; therefore (*hyp.* and E. xxviii. 1.) the figure $EBFC$ is a parallelogram, and (E. xxxiv. 1.) $EF=BC$.

First, let both the perpendiculars BE and CF fall without AD, so that both of them meet AD produced.

$$\left.\begin{aligned}\therefore\ AC^2&=DC^2+AD^2+2AD\times\overline{DF}\\ \text{and } BD^2&=AB^2+AD^2+2AD\times\overline{AE}\end{aligned}\right\}\text{(E. xii. 2.)}$$

$$\therefore\ AC^2+BD^2=AB^2+DC^2+2AD^2+2AD\times AE$$
$$+2AD\times DF.$$

But $2AD^2+2AD\times AE+2AD\times DF=2AD\times EF$ (E. i. 2.)

$$\therefore\ AC^2+BD^2=AB^2+DC^2+2AD\times EF:$$
$$\therefore\ AC^2+BD^2=AB^2+DC^2+2AD\times BC;$$

because, as hath been shewn, $EF=BC$.

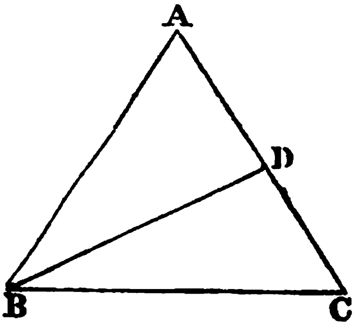
And, in like manner, may the proposition be demonstrated, by the help of E. xiii. 2. if one of the perpendiculars drawn from *B*, *C*, fall within *AD* the less of the two parallel sides.

Prop. XIII.

15. Theorem. *The square of the base of an isosceles triangle is the double of the rectangle contained by either side, and by the straight line intercepted between the perpendicular, let fall upon it from the opposite angle, and the extremity of the base.*

If the vertical angle of the isosceles triangle be a right angle, the proof of the proposition is manifestly deducible from E. xlvii. 1.

But let *ABC*, be an isosceles triangle, having its



vertical angle *A*, not a right angle: First, let *A* be an acute angle, and let *BD* be the perpendicular drawn from *B* to the opposite side *AC*:

Then, $\overline{BC}^2 = 2\,\overline{AC} \times \overline{CD}$.

For, since *BD* is perpendicular to *AC*,
therefore (E. xiii. 2.)

$$\therefore \overline{AB}^2 + 2\,AC \times CD = AC^2 + BC^2.$$

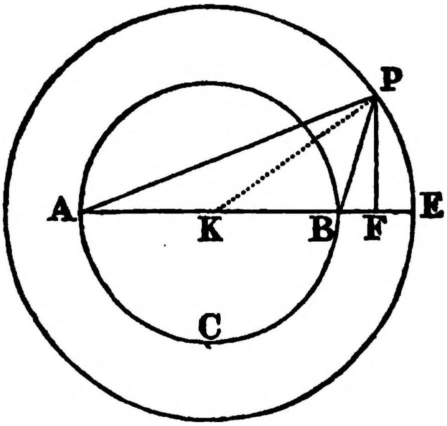
From these equals take away the equal squares (*hyp.*) AB^2 and AC^2, and there remains $BC^2 = 2\,AC \times CD$.

Secondly, if the $\angle A$ be obtuse, the proposition is manifestly deducible from E. xii. 2. and E. i. 2.

Prop. XIV.

16. Theorem. *If from any point, in the circumference of the greater of two given concentric circles, two straight lines be drawn to the extremities of any diameter of the less, their squares shall be, together, the double of the squares of the two semi-diameters of the two given circles.*

Let ACB, PDE, be two circles having a common centre K: and from any point P, in the circumference of the greater, let $\overline{PA}$, $\overline{PB}$, be drawn to the extremities A and B, of any diameter AKB, of the less circle: Then $PA^2+PB^2=2\overline{KA}^2+2\overline{KP}^2$, KA being a semi-diameter of the less, and KP a semi-diameter of the greater circle.



From P draw (E. xii. 1.) $\overline{PF}$ perpendicular to $\overline{AB}$; and, first, let PF fall without AB. And, because PF is perpendicular to AB,

$$\therefore PB^2 + 2\,\overline{BK} \times \overline{KF} = KP^2 + \overline{KB}^2 \text{ (E. xiii. 2.)}$$

$$\text{also, } PA^2 = \overline{KP}^2 + KA^2 + 2\,KA \times \overline{KF};$$

wherefore, since (E. xv. Def. 1.) $KB = KA$, if to the two former equals, the two latter be added, and if the equal rectangles, $2\,BK \times KF$, and $2\,KA \times KF$, be taken from the equal aggregates, it is manifest that

$$PA^2 + PB^2 = 2\,KA^2 + 2\,KP^2.$$

And, in like manner, the proposition may be demonstrated, when the perpendicular PF falls within $\overline{AB}$, the diameter of the lesser circle.

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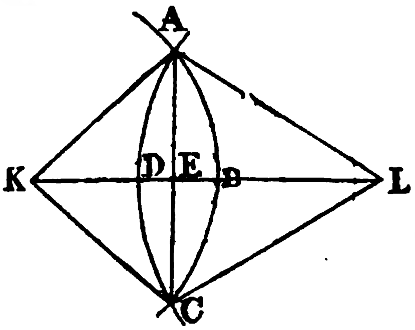
PART I.

BOOK III.

Prop. I.

1. **Theorem.** *If two circles cut each other, the straight line joining their two points of intersection is bisected, at right angles, by the straight line joining their centres.*

Let the circle *ABC*, of which the centre is *K*,



and the circle *ADC*, of which the centre is *L*, cut one another in the points *A*, *C*; and let *K*, *L*, and *A*, *C* be joined: $\overline{AC}$ is bisected, at right angles, in *E*, by $\overline{KL}$.

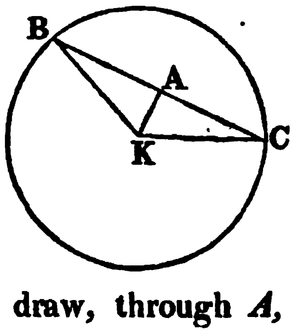
For, join *K*, *A*, and *K*, *C*, and *L*, *A*, and *L*, *C*: And since (E. Def. 15. 1.) $KA = KC$ and $LA = LC$, and that *KL* is common to the two $\triangle$s *KAL*, *KCL*, therefore (E. viii. 1.) the $\angle AKL = \angle CKL$. Again, since $AK = CK$, and that *KE* is common to the two $\triangle$s *AEK*, *CEK*, and, as hath been shewn, the $\angle AKE = \angle CKE$; therefore (E. iv. 1.) $AE = CE$, and the $\angle AEK = \angle CEK$, so that (E. Def. 10. 1.) each of these angles is a right angle. Wherefore, *KL* bisects *AC* at right angles.

2. Cor. Hence, if a trapezium have two of its adjacent sides equal to one another, and also its two remaining sides equal to one another, its diameters cut each other at right angles, and that which is the common base of two isosceles triangles is bisected by the other.

Prop. II.

3. Problem. ***Through a given point within a circle, which is not the centre, to draw a chord which shall be bisected in that point.***

Let *A* be a given point within the circle *BCD*:



It is required to draw, through *A*, a chord of the circle *BCD*, which shall be bisected in the point *A*.

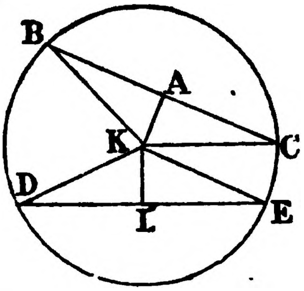
Find (E. i. 3.) the centre K of the circle BCD; join A, K; draw (E. xi. 1.) through A, the chord BAC perpendicular to KA: Then is BC bisected in the given point A.

For (*constr.*) KA, which is drawn through the centre K, cuts BC at right angles in A; therefore (E. iii. 3.) $BA = AC$.

Prop. III.

4. Theorem. *If two isosceles triangles be of equal altitudes, and the side of one be equal to the side of the other, their bases shall be equal.*

Let BKC, DKE be two isosceles triangles, having either of the equal sides, as BK, of the one, equal to either of the equal sides, as DK, of the other, and having, also, their altitudes, that is, the perpendiculars drawn from the vertex to the base in each, equal to one another: The base BC is equal, also, to the base DE.

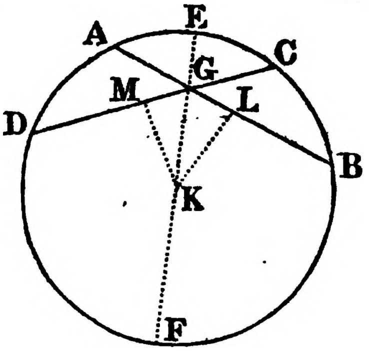


For the triangles being supposed to be so placed as to have their summits in the same point K, if they wholly coincide, it is manifest, that $BC = DE$; And, if they do not coincide, since (*hyp.*) $KB = KD = KE = KC$, a circle described from the centre K, at the distance *KB*, will pass through *D*, *E*, and *C*. From *K* as a centre, describe, therefore, the circle *BDEC*; and since (*hyp.*) the perpendicular *KA* is equal to the perpendicular *KL*, ∴ (E. xiv. 3.) *BC* = *DE*.

Prop. IV.

5. Theorem. *Any two chords of a circle, which cut a diameter in the same point, and at equal angles, are equal to one another.*

Let any two chords *AB*, *CD* of the circle *ADBC*



cut a diameter *EF* in the same point *G*, and make with it the ∠ *AGE* = ∠ *CGE*: Then *AB* = *CD*.

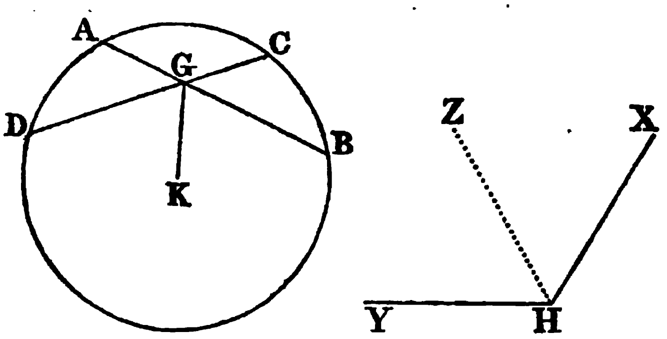
For, from *K*, the centre of the circle, draw (E. xii. 1.) $\overline{KL}$ perpendicular to *AB*, and $\overline{KM}$ perpendicular to *CD*. And since (*hyp.* and E. xv. 1.) the ∠ *LGK* = ∠ *MGK*, and (*constr.*) the angles at *L* and *M* are right angles, and that *KG* is common to the two △s *KLG*, *KMG*, therefore (E. xxvi. 1.) *KL* = *KM*; therefore (E. xiv. 3.) *AB* = *CD*.

Prop. V.

6. Problem. *Through a given point, within a*

given circle, to draw two equal chords, making with one another an angle equal to a given rectilineal angle.

Let *G* be a point in the circle *ADBC*, and



XHY a rectilineal angle: It is required to draw through *G* two equal chords of the circle *ADBC*, which shall make with one another an angle = ∠ *XHY*.

Find the centre *K* (E. i. 3.) of the circle *ADBC*, and join *K*, *G*; bisect (E. ix. 1.) the ∠ *XHY* by $\overline{ZH}$; at the point *G*, in *KG*, make (E. xxiii. 1.) the ∠s *KGB*, *KGD* each equal to the ∠ *ZHX* or *ZHY*, and produce *BG* and *DG* to meet the circumference in *A* and *C* respectively.

Then (*constr.*) the whole ∠ *BGD* = ∠ *XHY*; and since (*constr.*) the ∠ *KGB* = ∠ *KGD*; therefore (Supp. iv. 3.) $\overline{AB} = \overline{CD}$.

Prop. VI.

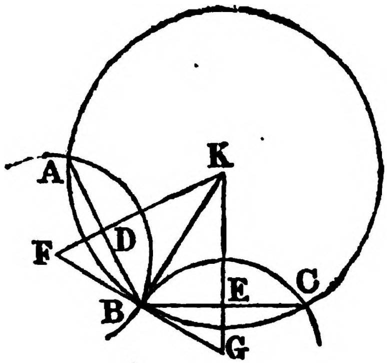
7. Theorem. *If the diameters of two circles are in the same straight line, and have a common extremity, the two circles shall touch one another.*

For since (*hyp.*) the two diameters are in the same straight line, it is manifest that a straight line drawn, from their common extremity, perpendicular to either of them will be perpendicular to the other, and therefore (E. xvi. 3. *Cor.*) will touch both the circles: The circles, therefore, (E. iii. Def. 3.) will touch one another: For it is plain that they cannot cut one another without also cutting the straight line that has been shewn to be their common tangent; which is impossible.

PROP. VII.

8. PROBLEM. *Three points being given in the circumference of a circle, and the middle point being equidistant from the other two, to describe two equal circles; which shall touch each other in the middle point, and which shall pass the one through one of the extreme points, and the other through the other extreme point.*

Let *A, B, C,* be three points in the circumference of the circle *ABC*, and let the middle



point, *B*, be equidistant from *A* and *C*: It is required to describe two equal circles, the one passing through *A* and the other through *C*, which shall touch one another in *B*.

Join *A*, *B*, and *B*, *C*; find (E. i. 3.) the centre *K* of the circle; from *K* draw (E. xii. 1.) $\overline{KD}$ perpendicular to $\overline{AB}$, and $\overline{KE}$ to $\overline{BC}$; join *K*, *B*; and through *B* draw (E. xi. 1.) $\overline{FBG}$ perpendicular to *KB* meeting *KD* and *KE*, produced in *F* and *G* respectively.

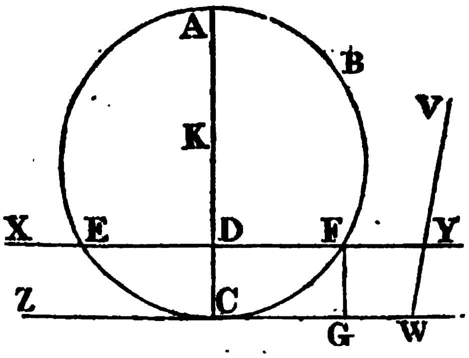
Then since (*hyp.*) *AB* = *BC*, therefore (E. xiv. 3.) *KD* = *KE*; and *KB* is common to the two △s *KDB* and *KEB*, and (*hyp. constr.* and E. iii. 3.) the side *DB* = the side *EB*; therefore (E. viii. 1.) the ∠*DKB* = ∠*EKB*.

And, since *KB* is common to the two △s *KBF*, *KBG*, and the ∠*FKB* = ∠*GKB*, and (*constr.*) the ∠*KBF* = ∠*KBG*, therefore (E. xxvi. 1.) *FB* = *GB*. If, therefore, from *F* and *G*, as centres, at the equal distances *FB*, *GB*, two equal circles be described, they will pass (*constr.* and Supp. iii. 1. *Cor.* 2.) the one through *A*, and the other through *C*, and (Supp. vi. 3.) they will touch one another in the point *B*.

PROP. VIII.

9. PROBLEM. *To draw a tangent to a circle, which shall be parallel to a given finite straight line.*

Let *ABC* be a circle, and *XY* a straight line: It is required to draw a straight line which shall



touch the circle *ABC*, and which shall be parallel to $\overline{XY}$.

Find (E. i. 3.) the centre *K* of the circle *ABC*; from *K* draw (E. xii. 1.) the diameter *AKC* perpendicular to *XY*; and from either of the extremities, as *C*, of *AC* draw (E. xi. 1.) $\overline{ZCW}$ perpendicular to *AC*.

Then since *ZWC* is perpendicular to *AC*, at its extremity *C*, it touches (E. xvi. 3. *Cor.*) the circle *ABC*: And since (*constr.*) the two ∠s *XDC*, *DCZ* are two right angles, therefore (E. xxviii. 1.) *ZW* is parallel to *XY*.

10. Cor. Hence a tangent may be drawn to a circle which shall make with a given straight line an angle equal to a given rectilineal angle.

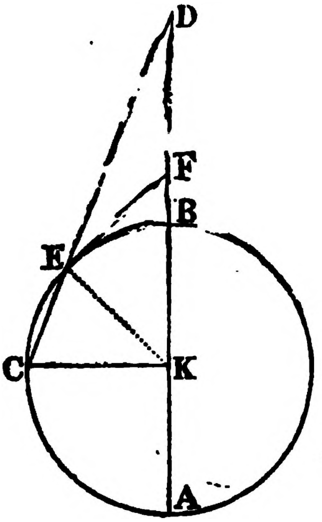
For let it be required to draw a tangent to the circle *ABC*, which shall make with a given straight line *VW* an angle equal to a given angle: Take any point *Y* in *VW*, and at the point *Y*, in $\overline{VY}$, make (E. xxiii. 1.) the ∠*VYX* equal to the given angle: If, then, the tangent *ZW* be drawn (Supp. viii. 3.) parallel to *YX*, it will make (E. xxix. 1.) the ∠*ZWV* = ∠*XYV*, which is equal to the given angle.

PROP. IX.

11. **PROBLEM.** *The diameter of a circle having been produced to a given point, to find in the part produced, a point from which, if a tangent be drawn to the circle, it shall be equal to the segment of the part produced, that is between the given point and the point found.*

Let the diameter *AB* of the circle *ABC* be produced to the given point *D*: It is required to find in *BD* a point from which if a tangent be drawn to the circle, it shall be equal to the part of *BD* which is between that point and *D*.

Find the centre *K* of the circle *ABC*; from *K* draw (E. xi. 1.) $\overline{KC}$ perpendicular to *AB*; join *D*, *C*, and let $\overline{DC}$ meet the circumference in *E*; join *K*, *E*; from *E* draw $\overline{EF}$ perpendicular to $\overline{KE}$ and let *EF* meet *BD* in *F*: Then is *F* the point which was to be found.

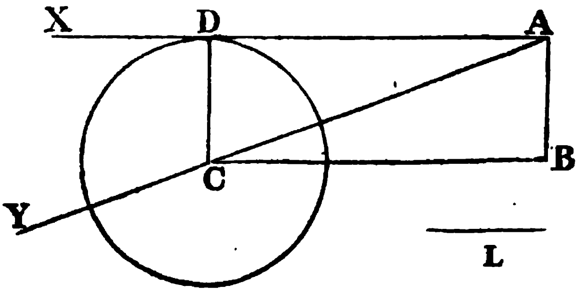


For (E. xiii. 1.) the $\angle$s KEC, KEF, FED are together equal to two right angles; as are, also, (E. xxxii. 1.) the three $\angle$s DCK, CKD, and KDC, of the $\triangle DKC$: But since (E. xv. Def. 1.) $KE=KC$, therefore (E. v. 1.) the $\angle KEC = \angle KCE$; and (*constr.*) the $\angle$s KEF, CKD are equal, each of them being a right angle; therefore the remaining $\angle FED$ is equal to the remaining $\angle KDC$ or FDE: therefore (E. vi. 1.) $FE=FD$; and since (*constr.*) EF is perpendicular to the semi-diameter KE, at its extremity E, therefore (E. xvi. 3.) FE touches the circle ABC. Q.E.F.

PROP. X.

12. PROBLEM. *To describe a circle which shall have a given semi-diameter and its centre in a given straight line, and shall also touch another straight line, inclined at a given angle to the former.*

Let AX and AY be two straight lines inclined to one another at a given angle; and let L be a finite straight line: It is required to describe a circle, which shall have its centre in AY, and its semi-diameter equal to L, and which shall touch AX.

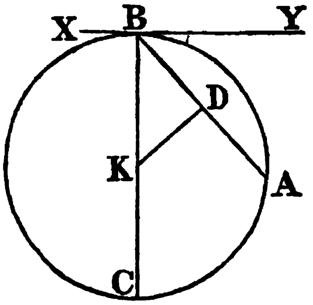


From the point *A*, in *AX*, draw (E. xi. 1.) *AB* perpendicular to *AX*, and make $AB=L$; through *B* draw (E. xxxi. 1.) *BC* parallel to *AX*, and through *C* draw *CD* parallel to *AB*: Wherefore, *DB* is a parallelogram; therefore (E. xxxiv. 1.) $DC=AB$; and since (*constr.*) the $\angle BAD$ is a right angle; therefore (E. xxix. 1.) the $\angle ADC$ is also a right angle: It is manifest, therefore, that a circle described from *C* as a centre, at the distance *CD*, will (E. xvi. 3. *Cor.*) touch *AX*; and its semi-diameter *CD* has been shewn to be equal to *AB*, which was made equal to the given straight line *L*. Q.E.F.

PROP. XI.

13. PROBLEM. *To describe a circle, the circumference of which shall pass through a given point, and touch a given straight line in another given point.*

Let *B* be a point in the straight line *XY*, and let *A* be any other point, without that line: It is required to describe a circle the circumference of which shall pass through *A* and touch *XY* in *B*.



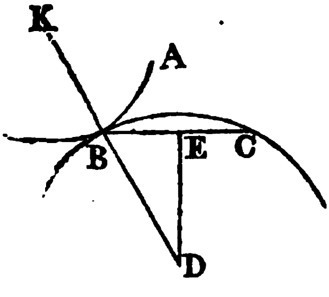
From *B* draw (E. xi. 1.) *BC* perpendicular to

XY; join A, B; bisect (E. x. 1.) $\overline{AB}$ in D, and from D draw $\overline{DK}$ perpendicular to AB; therefore (Supp. iii. 1. *Cor.* 2.) K is equidistant from A and B: It is manifest, therefore, that the circumference of a circle described from K as a centre, at the distance KB will pass through A; and since BY (*constr.*) is perpendicular to KB, the circle so described will (E. xvi. 3. *Cor.*) touch XY in B.

Prop. XII.

14. **Problem.** *To describe a circle, the circumference of which shall pass through a given point, and touch a given circle in another given point; the two points not lying in a tangent to the circle.*

Let B be a point in the circle AB, and C any other point, which is not in a tangent to the circle at B: It is required to describe a circle, the circumference of which shall pass through C and touch the circle AB in B.

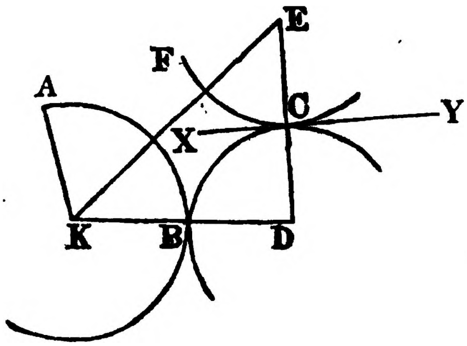


Find (E. i. 3.) the centre K of the circle AB; join C, B and K, B; bisect (E. x. 1.) CB in E; draw (E. xi. 1.) ED perpendicular to CB, and let ED meet KB, produced if necessary, in D: Then, since (Supp. iii. 1. *Cor.* 2.) the point *D* is equidistant from *B* and *C*, the circumference of a circle described from *D* as a centre, at the distance *DB*, will pass through *C*, and (Supp. vi. 3.) it will touch the circle *AB* in *B*.

Prop. XIII.

15. **Problem.** *To describe a circle, which shall touch a given straight line in a given point, and also touch a given circle.*

Let *AB* be a circle, and let *C* be a point in the straight line *XY*: It is required to describe a circle which shall touch *XY* in *C*, and which shall also touch the circle *AB*.



Through *C* draw (E. xi. 1.) $\overline{ECD}$ perpendicular to *XY*; find (E. i. 3.) the centre *K* of the circle *AB*, and draw any semi-diameter of it at *KA*; make (E. iii. 1.) *CE*=*KA*, and join *E*, *K*; at the point *K*, in *EK*, make (E. xxiii. 1.) the $\angle EKD = \angle KED$, and let $\overline{KD}$ meet *ECD* in *D*: Then, since (*constr.*) the $\angle DEK = \angle DKE$, therefore (E. vi. 1.) *DE*=*DK*; and *CE*=*BK*, for *CE* was made equal to *KA*, and (E. xv. Def. 1.) *KA*=*KB*; therefore, the remainder

DC is equal to the remainder *DB*; therefore, a circle described from *D*, as a centre, at the distance *DC*, will pass through *B*; and (E. xvi. 3. *Cor*. and *constr.*) it will touch $\overline{XY}$ in *C*, and (Supp. vi. 3.) it will also touch the circle *AB* in *B*.

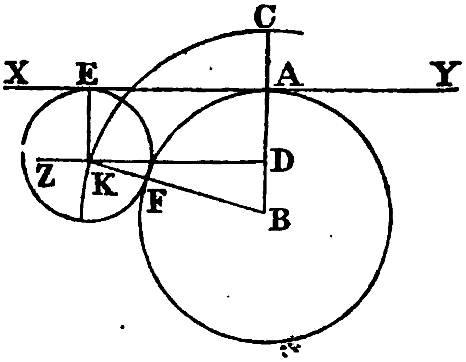
16. Cor. It is manifest that, in the same manner, a circle may be described which shall touch a given circle in a given point, and which shall, also, touch another given circle.

For, if a straight line be drawn at right angles to the diameter of the given circle that passes through the given point, then the solution of this latter problem is, evidently, reduced to that of the former.

Prop. XIV.

17. Problem. *To describe two circles, each having a given semi-diameter, which shall touch the same given straight line, both on the same side of it, and shall also touch each other.*

Let *XY* be a straight line, of indefinite length: It is required to describe two circles, each having a given semi-diameter, which shall touch $\overline{XY}$, and also touch one another.



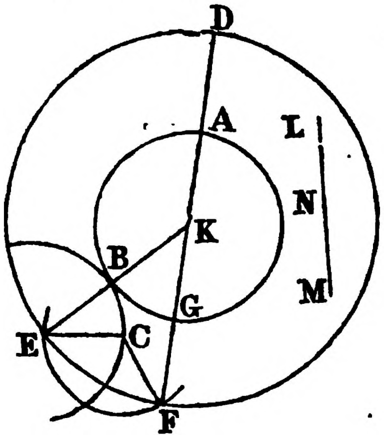
Take any point A in XY; through A draw (E. xi. 1.) $\overline{CB}$ perpendicular to XY, and make AB equal to the given semi-diameter of one of the circles, and AC equal to the given semi-diameter of the other; from B, as a centre, at the distance BA describe the circle AF; from AB, produced, if necessary, cut off (E. iii. 1.) $AD = AC$; through D draw (E. xxxi. 1.) $\overline{DZ}$ parallel to XY; from B as a centre, at the distance BC describe a circle, and let its circumference cut DZ in K; through K draw KE parallel to AD, and join K, B; therefore the figure $AEKD$ is a parallelogram, and (E. xxxiv. 1.) $KE = DA$ or AC; also (*constr.* and E. xv. Def. 1.) $BC = BK$; and $BA = BF$; therefore the remainder $AC =$ the remainder FK, and it has been shewn that $AC = KE$; $\therefore$ $KE = KF$; therefore, a circle described from K as a centre, at the distance KE, will pass through F and (Supp. vi. 3.) will touch the circle AF, which circle (*constr.* and E. xvi. 3. *Cor.*) touches XY; and since (*constr.*) the $\angle DAE$ is a right angle, and that KE was drawn parallel to DA, therefore (E. xxix. 1.) the $\angle KEA$ is a right angle; therefore (E. xvi. 3. *Cor.*) the circle EF, also, touches XY; and its semi-diameter KE has been shewn to be equal to AD, which was made equal to the given semi-diameter. Q.E.F.

PROP. XV.

18. PROBLEM. *To describe two equal circles, each having its diameter equal to a given straight line, each touching a given circle, and each also passing through a given point without that circle: The given*

straight line being greater than the shortest distance, between the given point and the circumference of the given circle.

Let *ABG* be a circle, *C* a point without it,



and *LM* a given straight line greater than the shortest distance between *C* and the circumference of *ABG*: It is required to describe two equal circles, each having its diameter equal to *LM*, each of them touching *ABG* and each passing through *C*.

Bisect (E. x. 1.) *LM* in *N*; take any point *A* in $\overset{\frown}{ABG}$; find (E. i. 3.) the centre *K* of *ABG*, and join *K*, *A*; produce *KA* to *D* and make *AD*=*NL* or *NM*; from *K* as a centre, at the distance *KD*, describe the circle *DEF*, and from the centre *C* at a distance equal to *NL* or *NM*, describe the circle *EF* and let $\overset{\frown}{EF}$ cut $\overset{\frown}{DEF}$ in the points *E*, and *F*; join *E*, *K*, and *F*, *K*, and let *EK* and *FK* meet $\overset{\frown}{ABG}$ in the points *B* and *G*; join, also, *C*, *E* and *C*, *F*.

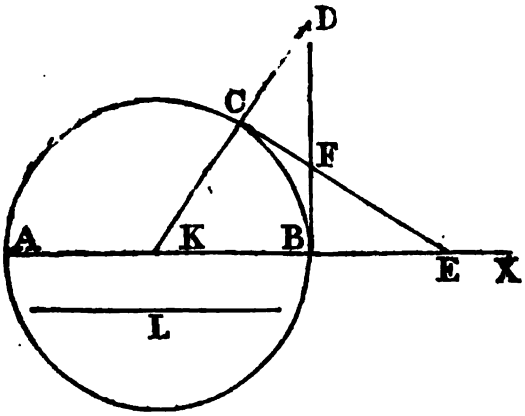
Then, it is manifest, (from the *constr*. and E. xv. Def. 1.) that *EB*, *EC*, *FC* and *FG* are each of them equal to *LN*, the half of the given straight line *LM*; therefore, the two equal circles described from the centres *E* and *F*, at the equal distances *EC* and *FC*, will each of them have its diameter equal to *LM*, will each of them pass through the given point *C*, and (Supp. vi. 3.) will touch the given circle *ABG* in *B* and *G*.

19. Cor. In the same manner two equal circles may be described, each of them touching two given concentric circles, and each passing through a given point situated between the circumferences of those two given circles.

Prop. XVI.

20. Problem. *To find a point in the diameter, produced, of a given circle, from which, if a tangent be drawn to the circle, it shall be equal to a given straight line.*

Let *AB* be a diameter of the given circle *ABC*,



and let *L* be a finite straight line: It is required to find a point in *AB*, produced, from which if a tangent be drawn to the circle *ABC*, it shall be equal to *L*.

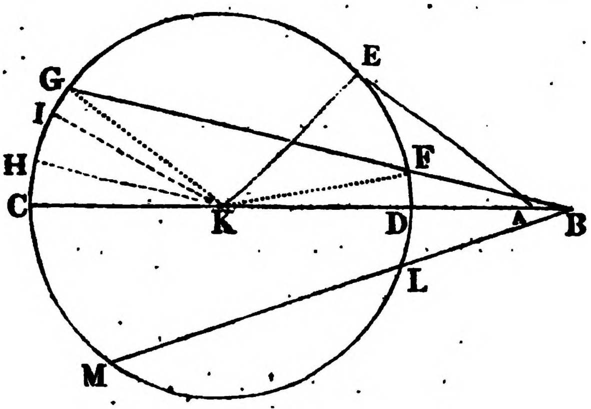
Produce *AB*, towards *X*; and from *B* draw (E. xi. 1.) $\overline{BD}$ perpendicular to $\overline{AB}$ and make $BD=\overline{L}$; find (E. i. 3.) the centre *K* of the circle *ABC*, and join *K*, *D*; let $\overline{KD}$ cut $\overset{\frown}{ABC}$ in *C*; from *KX* cut off (E. iii. 1.) $KE=KD$: Then is *E* the point which was to be found.

For, join *E*, *C*: And since (*constr.* and E. xv. Def. 1.) the two sides *DK*, *KB*, are equal to the two sides *EK*, *KC*, and that the $\angle K$ is common to the two △s *DBK*, *ECK*, therefore (E. iv. 1.) $EC=BD$ and the $\angle ECK=\angle DBK$; but (*constr.*) $BD=L$, and the $\angle DBK$ is a right angle; therefore $EC=L$, and the $\angle ECK$ is a right angle; therefore (E. xvi. 1. *Cor.*) *EC* is a tangent to the circle *ABC*.

PROP. XVII.

21. THEOREM. *If the straight line, drawn from a point in the produced diameter of a circle to the convex circumference be equal to the half of the diameter, the angle at the centre, subtended by the* concave *circumference included between the diameter and the line* so *drawn, is the triple of the angle, at the centre, subtended by the* convex *circumference included between the same two lines.*

Let *CDE* be a circle, of which *K* is the centre, and *CDB*, a produced diameter; and let $\overline{AE}$, which touches the circumference in *E*, or $\overline{BF}$, a part of $\overline{BFG}$, which cuts it, be equal to the semi-



diameter of the circle: Then K, E, and K, F, and K, G, having been joined, the $\angle EKC = 3\angle EKD$; and the $\angle GKC = 3\angle FKD$.

For, first, since AE touches the circle, the $\angle AEK$ (E. xviii. 3.) is a right angle; therefore (E. xxxii. 1.) the $\angle EAK + \angle EKA = \angle KEA$; and (*hyp.*) $EA = EK$; therefore (E. v. 1.) the $\angle EAK = \angle EKA$; therefore the $\angle KEA = 2\angle EKA$; but (E. xxxii. 1.) the $\angle EKC = \angle KEA + \angle EKA$; therefore the $\angle EKC = 3\angle EKA$.

Secondly, since (*hyp.* and E. v. 1.) the $\angle FBK = \angle FKB$, and (E. xxxii. 1.) the $\angle GFK = \angle FKB + \angle FBK$, therefore the $\angle GFK = 2\angle FKB$: But (E. xv. Def. 1.) $KF = KG$; therefore the $\angle KFG = \angle KGF$; therefore the $\angle KGF = 2\angle FKD$; and (E. xxxii. 1.) the $\angle GKC = \angle KGB + \angle GBK = \angle KGF + \angle FKD$; therefore the $\angle GKC = 3\angle FKD$.

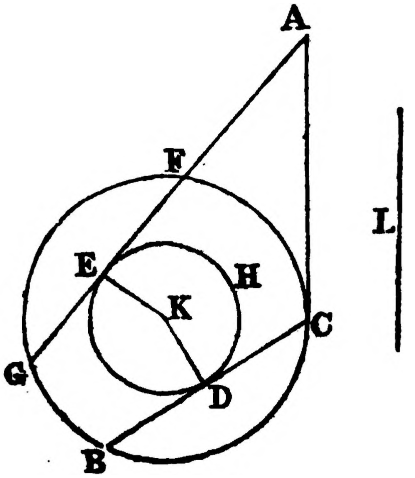
22. Cor. Hence, if a straight line could be drawn from any point in the curve of a semi-circle, to meet the diameter produced, so that the part of the line without the curve should be equal to the semi-diameter, any angle might be trisected.

For, let *GKC* be any given angle; from *K* as a centre, at any distance *KC*, describe the circle *CGD*, and produce the diameter *CD*: Then, if from *G*, $\overline{GFB}$ could be drawn to meet *CD* produced in *B*, so that the part of it, *FB*, without the circle, should be equal to the semi-diameter *KC*, it is manifest from the proposition, that the ∠*GBC* is the third part of the ∠*GKC*: If, therefore, at the point *K* in $\overline{CK}$, the ∠*CKH* were made (E. xxiii. 1.) equal to the ∠*CBG*, and if, also, at the point *K* in $\overline{HK}$ the ∠*HKI* were made equal to the same ∠*CBG*, it is plain that the given ∠*GKC* would thereby be divided into three equal parts.

PROP. XVIII.

23. PROBLEM. *Through a given point, either within, or without a given circle, to draw a straight line, so that the part of it within the circle shall be equal to a given finite straight line, which is not greater than the diameter.*

Let *A* be the given point, and *L* the given finite



straight line, not greater than the diameter of *FBC* the given circle: It is required to draw through *A* a straight line cutting *FBC*, so that the part of it within the circle shall be equal to *L*.

Find (E. i. 3.) the centre *K* of the given circle; and if *L* be equal to its diameter, let *A*, *K* be joined, and it is manifest that $\overline{AK}$ produced will be the straight line which was to be drawn.

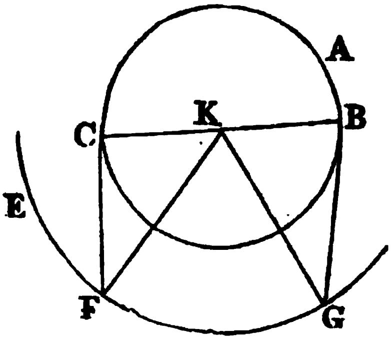
But if *L* be less than the diameter, take any point *B* in $\overset{\frown}{FGBC}$; from *B* as a centre, at a distance equal to *L*, describe a circle cutting $\overset{\frown}{FGBC}$ in *C*, and join *B*, *C*; therefore (E. xv. Def. 1.) $BC = L$: From *K* draw (E. xii. 1.) *KD* perpendicular to *BC*; from the centre *K* at the distance *KD*, describe the circle *ED*; from the point *A* draw (E. xvii. 3.) $\overline{AE}$ touching the circle *ED* in *E*, and let *AE*, produced, meet the circumference in *F* and *G*: Then $FG = L$.

For (E. xv. Def. 1.) $KE = KD$, and since *AE* touches the circle *ED* in *E*, the ∠*AEK* is (E. xviii. 3.) a right angle, as is also (*constr.*) the ∠*KDB*; therefore (E. xiv. 3.) $FG = CB$; but *CB* was made equal to *L*: ∴ $FG = L$.

PROP. XIX.

24. THEOREM. *If, from any two points in the circumference of the greater of two given concentric circles, two straight lines be drawn so as to touch the less circle, they shall be equal to one another.*

Let *F*, *G*, be any two points in the circumference *EFG* of the greater of two circles, *EFG*, *ABC*, which have a common centre *K*: Two straight lines drawn from F and G so as to touch the less circle ABC shall be equal to one another.



For, draw (E. xvii. 3.) from F and G, $\overline{FC}$ and $\overline{GB}$ touching ABC in C and B respectively; and join K, C and K, F and K, G and K, B; therefore (E. xviii. 3.) the ∠s KCF and KBG are right angles; therefore (E. xlvii. 1.)

$$\overline{KF}^2 = \overline{KC}^2 + \overline{CF}^2;$$

$$\text{and } \overline{KG}^2 = \overline{KB}^2 + \overline{BG}^2:$$

But (E. xv. Def. 1.) $KF = KG$, and $KC = KB$; therefore $KC^2 + CF^2 = KB^2 + BG^2$; take away the equal squares, KC^2, and KB^2, and there remains $CF^2 = BG^2$; $\therefore$ $CF = BG$.

25. Cor. 1. In the same manner it may be shewn, that if two straight lines be drawn from any the same point so as to touch a given circle, they shall be equal to one another; and therefore, (E. viii. 1.) the straight line joining that point and the centre, bisects the angle contained by the two equal tangents.

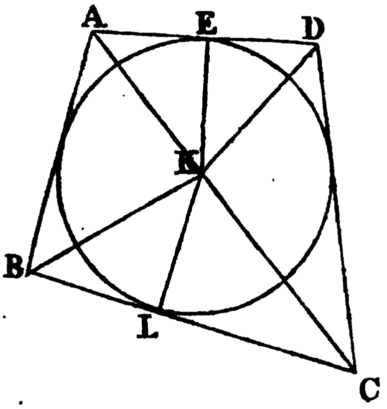
26. Cor. 2. If two circles touch one another and also touch a given straight line, which does not pass through their common point of contact, a straight

line that touches both the circles in their common point of contact shall bisect that other tangent straight line.

PROP. XX.

27. THEOREM. *If a quadrilateral rectilineal figure be described about a circle, the angles subtended, at the centre of the circle, by any two opposite sides of the figure, are, together, equal to two right angles.*

Let the quadrilateral figure *ABCD* be described



about the circle *EFLM*, of which the centre is *K*; the angles subtended at *K*, by the two opposite sides *AD*, *BC*, or by *AB*, *DC*, are together, equal to two right angles.

For join *K*, *A*, and *K*, *B*, and *K*, *C*, and *K*, *D*: Then, because (E. xxxii. 1. *Cor.*) the four interior ∠s *A*, *B*, *C*, *D*, of the figure *ABCD*, are equal to four right angles, and that (Supp. xix. 3. *Cor.* 1.) they are bisected by $\overline{KA}$, $\overline{KB}$, $\overline{KC}$ and $\overline{KD}$, respectively, therefore the ∠s *KAD*, *KDA*, *KBC*, *KCB* are, together, equal to two right angles; but (E. xxxii. 1.)

those angles, together with the ∠s *AKD*, *BKC*, being all the angles of the two △s *AKD*, *BKC*, are equal to four right angles; therefore the ∠s *AKD*, *BKC*, are, together, equal to two right angles; therefore (E. xv. 1. *Cor.* 2.) the ∠s *AKB*, *DKC*, are, also, taken together, equal to two right angles.

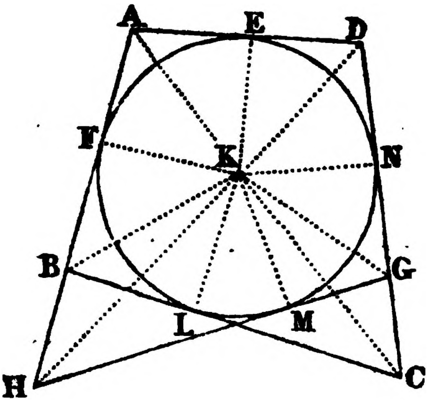
28. Cor. If two of the sides as *AD*, *BC*, of the quadrilateral figure described about the circle *EFL*, touch the circle at the extremities of a diameter, the angles subtended at the centre *K*, by each of the two remaining sides, shall be right angles.

For then since (E. xviii. 3. and E. xxviii. 1.) $\overline{AD}$ is parallel to $\overline{BC}$; therefore (E. xxix. 1.) the ∠ *EAB* + ∠ *ABL* = two right angles; and (Supp. xix. 3. *Cor.* 1.) the ∠ *EAB* is double of the ∠ *BAK*; in the same manner the ∠ *ABL* may be shewn to be double of the ∠ *ABK*; but it has been proved that the ∠ *EAB*+∠ *ABL*=two right angles; therefore the ∠*KAB*+∠ *KBA*=a right angle; ∴ (E. xxxii. 1.) the ∠ *AKB* is a right angle: But (Supp. xx. 3.) the ∠ *AKB*+ ∠ *DKC*=two right angles; therefore the ∠ *DKC* is, also, a right angle.

Prop. XXI.

29. Theorem. *If two given straight lines touch a circle, and if any number of other tangents be drawn, all on the same side of the centre, and all terminated by the two given tangents, the angles which they subtend, at the centre of the circle, shall be equal to one another.*

Let the two straight lines *AH*, *DC* touch the circle *EFLM*, and let $\overline{BC}$ and $\overline{GH}$ be any other



tangents of the circle, both on the same side of the centre *K*, and both terminated by $\overline{AH}$ and $\overline{DC}$: Then $\overline{BC}$ and $\overline{GH}$ subtend equal angles at the centre.

For draw (E. xvii. 3.) any other tangent to the circle, on the contrary side of the centre, as *DEA*, terminated in *A* and *D*, by *AH* and *DC*; and draw $\overline{KA}$, $\overline{KB}$, $\overline{KH}$, and $\overline{KC}$, $\overline{KG}$ and $\overline{KD}$: And because *ABCD*, *AHGD*, are quadrilateral figures described about the circle, therefore (Supp. xx. 3.) the $\angle AKD + \angle BKC =$ two right angles; and, the $\angle AKD + \angle HKG =$ two right angles; therefore the $\angle BKC = \angle HKG$; *i. e.* the angle subtended at the centre by the tangent *BC* is equal to the angle subtended at the centre by the tangent *HG*.

30. Cor. The two segments, which any two tangents, so drawn, cut off from the two given tangents, also subtend equal angles, at the centre of the circle.

Let *BH* and *GC* be the segments cut off from the tangents *AH* and *DC*, by the two tangents *BC* and

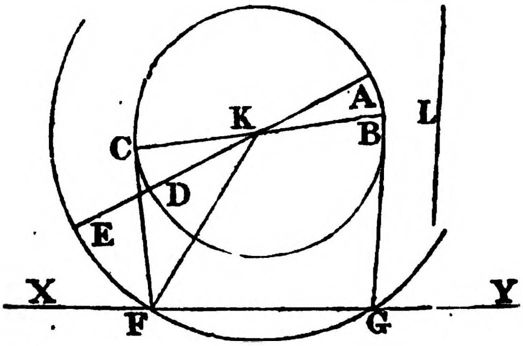
GH: They subtend equal ∠s *BKH*, *GKC* at the centre *K*.

For it has been shewn that the ∠ *BKC* = ∠ *HKG*; from these equals take away the common ∠ *HKC*, and there remains the ∠ *BKH* = ∠ *GKC*.

Prop. XXII.

31. Problem. *To draw a tangent to a given circle, such that its segment, contained between the point of contact, and an indefinite straight line, given in position, shall be equal to a given finite straight line.*

Let *ABC* be the given circle, *L* a finite straight line, and *XY* an indefinite straight line given in position: It is required to draw a tangent to *ABC* so that its segment between the point of contact and *XY* shall be equal to *L*.

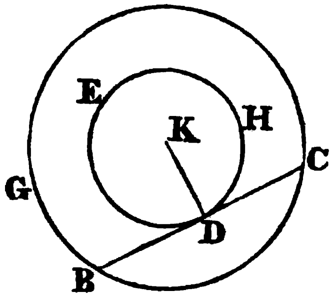


Find the centre *K* (E. i. 3.) of the given circle, and take any diameter of it, as *AKD*; in *AD* produced find (Supp. xvi. 3.) a point *E* from which if a tangent be drawn to the given circle *ABC* it shall be equal to *L*; from *K* as a centre, at the distance *KE*, describe the circle *EFG*, and let it meet, or cut, $\overline{XY}$ in *F*; from *F* draw (E. xvii. 3.) $\overline{FC}$ to touch the circle *ABC* in *C*: And since (Supp. xix. 3.) *FC* is equal to the tangent which can be drawn from *E*, and which (*constr.*) is itself equal to *L*, it is manifest that $CF = L$; *i. e.* the segment of the tangent between the point of contact *C* and *XY* is equal to the given straight line *L*.

Prop. XXIII.

32. Theorem. *If a straight line touch the interior of two concentric circles, and be terminated both ways by the circumference of the outer circle, it shall be bisected in the point of contact.*

Let *GBC, EDH* be two circles having a common



centre *K*, and let $\overline{BC}$ touch the interior circle *EDH* in *D*: Then is *BC* bisected in *D*.

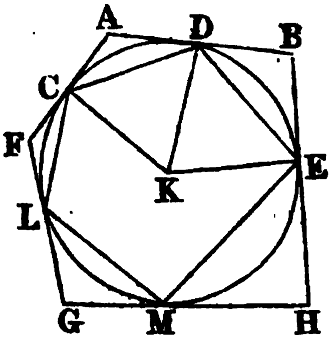
For join *K, D*: And, because *BC* touches *EDH* in *D*, the ∠s *KDC, KDB* (E. xviii. 3.) are right angles; therefore (E. iii. 3.) *BC* is bisected in *D*.

Prop. XXIV.

33. Theorem. *If a polygon be described about a circle, the straight lines joining the several points of contact will contain a polygon of the same number of*

angles as the former; and any two adjacent angles of the circumscribed figure shall be, together, the double of that angle, of the inscribed figure, which lies between them.

Let the sides of the polygon $AFGHB$ touch the



circle $CLMED$, in the several points C, L, M, E and D, and let these points be joined: Then it is manifest, that the polygon $DCLME$ has the same number of angles as $AFGHB$; and, further, any two adjacent $\angle$s A and B of the polygon $AFGHB$, are, together, the double of the intermediate $\angle\ CDE$, of the inscribed figure.

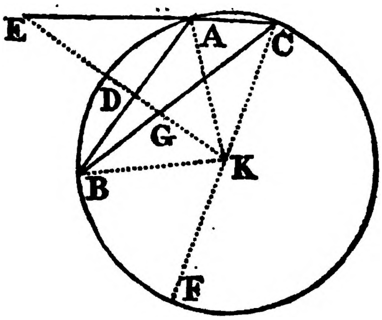
For, find (E. i. 3.) the centre K of the circle $DCLME$, and join K, C and K, D and K, E: The four interior angles of the quadrilateral figure $ACKD$ are (E. xxxii. 1. *Cor.* 1.) equal to four right angles; and (*hyp.* and E. xviii. 3.) the $\angle$s ACK and ADK are right angles; therefore the $\angle\ DAC + \angle\ CKD$ is equal to two right angles, as are also (E. xxxii. 1.) the three angles of the isosceles $\triangle\ CKD$; $\therefore\ \angle\ DAC + \angle\ CKD = \angle\ DCK + \angle\ CKD + \angle\ KDC$; take away the common $\angle\ CKD$, and there remains the $\angle\ DAC$ equal to the two $\angle$s DCK, KDC, or to the double of the $\angle\ KDC$; because (E. xv. Def. 1. and

v. 1.) the $\angle DCK = \angle KDC$: And, in the same manner, it may be shewn that the $\angle B$ is the double of the $\angle KDE$; therefore the $\angle A + \angle B$ is the double of the whole $\angle CDE$.

PROP. XXV.

34. THEOREM. *If from any given point, in the circumference of a circle, two straight lines be drawn to the extremities of a given chord, the angle which the one makes with any perpendicular to the chord, shall be equal to the angle which the other makes with the diameter of the circle that passes through the given point.*

Let C be a point in the circumference of the circle $ABFC$; let AB be a chord; let C, A and C, B be joined; let K be the centre of the circle, and CKF a diameter passing through C; and let KD, drawn perpendicular to AB, meet CB in G, and CA, produced, in E: Then, the $\angle KEC = \angle BCF$, and the $\angle EGB = \angle ECF$.



For (E. xxxii. 1.) $\angle AEK + \angle AKE = \angle CAK$: But (*demonstr.* of E. iii. 3. and *constr.*)

the $\angle AKE = \frac{1}{2} \angle AKB$;

And (E. xx. 3.) the $\angle ACB = \frac{1}{2} \angle AKB$;

$$\therefore \angle ACB = \angle AKE.$$

Also (E. xv. Def. 1. and E. v. 1.)

$$\angle CAK = \angle ACK;$$

$$\therefore \angle AEK + \angle ACB = \angle ACK.$$

Take from both the $\angle ACB$ and there remains

$$\angle AEK \text{ or } KEC = \angle BCF.$$

Again (E. xxxii. 1.) the

$$\angle EGB = \angle ECG + \angle CEG:$$

And it has been shewn that the $\angle CEG = \angle BCF$;

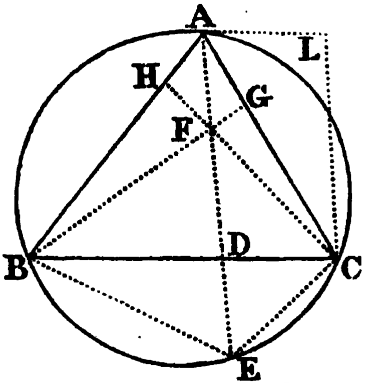
$$\therefore \angle EGB = \angle ECB + \angle BCF,$$

$$i.\,e.\ \triangle EGB = \angle ECF.$$

PROP. XXVI.

35. THEOREM. *The perpendiculars let fall from the three angles of any triangle upon the opposite sides, intersect each other in the same point.*

Let ABC be a triangle; the perpendiculars let fall



from the three $\angle$s A, B, C, on the sides opposite to them, intersect each other in the same point.

For draw (E. xii. 1.) $\overline{AD}$ perpendicular to BC; about the $\triangle ABC$ describe (Supp. v. 1. *Cor.*) the circle *ABC*, and produce *AD* to meet the circumference in *E*; from *DA*, produced if necessary, cut off *DF*=*DE*; join *B*, *E* and *C*, *E*; also join *B*, *F* and *C*, *F*; and let *BF* and *CF* produced meet *AC*, and *AB*, in *G* and *H* respectively.

And since *CD* is common to the two △s *CDF*, *CDE*, and that (*constr.*) *DF*= *DE*, and the ∠ *CDF* = ∠ *CDE*; therefore (E. iv. 1.) the ∠ *FCD*= ∠ *DCE* or *BCE*; but (E. xxi. 3.) the ∠ *BAE*= ∠ *BCE*; therefore the ∠ *BAE* or *HAE* = ∠ *FCD*; and (E. xv. 1.) the ∠ *AFH*, of the △ *AHF*, is also equal to the ∠ *DFC*, of the △ *CDF*; therefore (Supp. xxvi. 1.) the ∠ *AHF*= ∠ *FDC*, which (*constr.*) is a right angle; therefore the ∠ *CHA* is a right angle; *i.e.* *CFH* is perpendicular to *AB*; and, in the same manner, it may be shewn that *BFG* is perpendicular to *AC*: Whence it is manifest, that the three perpendiculars cut each other in the common point *F*; for (E. xvii. 1.) there cannot be drawn, from the same point, two different straight lines both of them perpendicular to the same straight line.

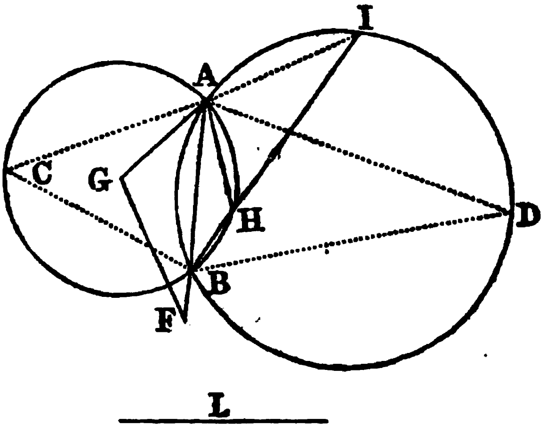
36. Cor. The part of any of the three perpendiculars, let fall from the three angles of a triangle, on the opposite sides, that is between their common intersection and the circumference of the circle described about the triangle, is bisected by the side to which it is perpendicular.

Prop. XXVII.

37. Problem. *From either of the two given points in which two given circles intersect each other, to draw a chord cutting the one circumference, and meeting the other, such that the part of it, contained*

between the two circumferences, shall be equal to a given finite straight line.

Let the two given circles *ABC*, *ABD*, cut one



another in the points *A* and *B*, and let *L* be a given finite straight line: From either of the two given points, as *B*, it is required to draw a straight line cutting either of the circumferences, as that of *ABC*, and meeting the other circumference, so that the part of it contained between the circumferences, shall be equal to *L*.

Take any point *C* in the circumference of *ABC*, and any point *D* in the circumference of *ABD*; join *A*, *B*, and *A*, *C*, and *A*, *D*, and *B*, *C* and *B*, *D*; in *AB*, produced if necessary, take $AF = L$; at the point *A*, in *AF*, make (E. xxiii. 1.) the $\angle FAG = \angle ACB$, and at the point *F*, make the $\angle AFG = \angle ADB$, and let $\overline{AG}$ and $\overline{FG}$ meet in *G*. In the circle *ABC* place $\overline{AH} = AG$; join *B*, *H*, and produce *BH* to meet the circumference of *ADB* in *I*: Then is $HI = L$.

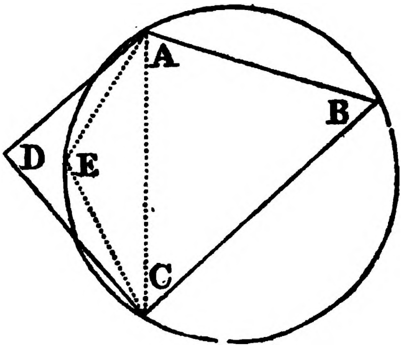
For, join *A*, *I*: And, since (E. xxii. 3.) the

$\angle AHB + \angle ACB =$ two right angles; and that (E. xiii. 1.) the $\angle AHB + \angle AHI =$ two right angles; therefore the $\angle AHI = \angle ACB$; but (*constr.*) the $\angle ACB = \angle FAG$; therefore the $\angle AHI = \angle FAG$; and (*constr.*) the $\angle AFG = \angle ADB$, which (E. xxi. 3.) $= \angle AIH$; therefore the $\angle AFG = \angle AIB$; and (*constr.*) the side AH, of the $\triangle HAI$, is equal to the side AG, of the $\triangle AGF$; therefore (E. xxvi. 1.) $HI = AF$; and (*constr.*) $AF = L$; $\therefore$ $HI = L$. Q.E.F.

PROP. XXVIII.

38. **THEOREM.** ***If two opposite angles of a quadrilateral figure be together equal to two right angles, a circle may be described about it.***

Let any two opposite angles, as the $\angle$s ABC, ADC, of the quadrilateral figure $ABCD$, be together equal to two right angles: A circle may be described about the trapezium $ABCD$.



For, join A, C; and (Supp. v. 1. *Cor.*) about the $\triangle ABC$ describe a circle: Its circumference shall pass through the point D. If not, let it pass otherwise, so that, first, the point D is without the circle

ABC, described about the $\triangle$ *ABC*; take any point *E* in the circumference of the circle and within the $\triangle$ *ADC*; and join *A*, *E* and *C*, *E*: Then, since *ABCE* is a quadrilateral figure inscribed in a circle the $\angle ABC + \angle AEC =$ two right angles; and (*hyp.*) the $\angle ABC + ADC =$ two right angles; therefore the $\angle AEC = \angle ADC$, which (E. xxi. 1.) is absurd. Wherefore the point *D* is not without the circle *ABC*; and in the same manner it may be shewn that the point *D* is not within the circle *ABC*; therefore, the circumference of the circle *ABC* passes through the point *D*, and is, therefore, a circle described about the four-sided figure *ABCD*.

PROP. XXIX.

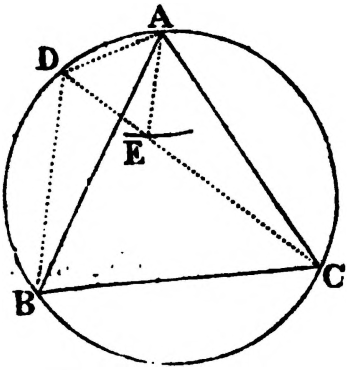
39. THEOREM. *A circle cannot be described about a rhombus, nor about any other parallelogram which is not rectangular.*

For (E. xxxiv. 1.) the opposite angles of a parallelogram are equal to one another; and (E. xxii. 3.) if a circle could be described about it, the two opposite angles would, together, be equal to two right angles; therefore, since these angles are equal, they would be each of them a right angle; but (E. xxxii. Def. 1.) the angles of a rhombus, which (E. xxxii. Def. 1. and Supp. xviii. 1.) is a parallelogram, are not right angles; therefore a circle cannot be described about a rhombus, nor about any other parallelogram, which has not its opposite angles right angles, that is (Supp. xix. 1.) which is not rectangular.

PROP. XXX.

40. THEOREM.. *If from any point, in the circumference of a given circle, straight lines be drawn to the three angles of an inscribed equilateral triangle, the greatest of them shall be equal to the aggregate of the two less.*

Let the equilateral △ *ABC* be inscribed in the circle *ADBC*, and from any point *D* in the circumference, let there be drawn to the three angular points *A*, *B*, *C*, the straight lines *DA*, *DB*, *DC*, of which *DC* is the greatest: Then *DC* = *DA* + *DB*.



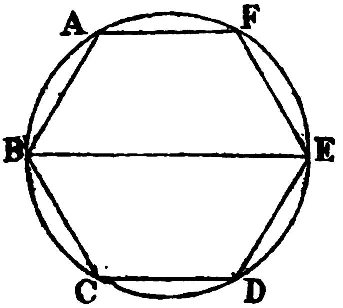
From the centre *A*, at the distance *AD*, describe a circle cutting *DC* in *E*, and join *A*, *E*; therefore (E. xv. Def. 1.) *AD* = *AE*; therefore (E. v. 1.) the ∠ *ADE* = ∠ *AED*; also (E. xxi. 3.) the ∠ *ADC* or *ADE* = ∠ *ABC*; and (*hyp*. and E. v. 1.) the ∠ *ABC* = ∠ *ACB*; therefore (Supp. xxvi. 1.) the ∠ *DAE* = ∠ *BAC*; therefore the △ *ADE* is equiangular; therefore (E. vi. 1.) *AD* = *DE*.

Again, since (E. xxii. 3.) the ∠ *ACB* + ∠ *ADB* = two right angles, and (E. xiii. 1.) the ∠ *AED* + ∠ *AEC* = two right angles, and that the ∠ *AED* has been shewn to be equal to the $\angle ACB$; therefore the $\angle AEC = \angle ADB$; also (E. xxi. 3.) the $\angle ACD$ or $ACE = \angle ABD$; and (*hyp.*) the side AC, of the $\triangle AEC$, is equal to the side AB of the $\triangle ADB$; therefore (E. xxvi. 1.) $EC = DB$: And DA has been proved to be equal to DE; therefore $DE+EC=DA+DB$; that is, $DC=DA+DB$.

Prop. XXXI.

41. Theorem. *The first, third, fifth, &c. angles of any polygon, of an even number of sides, which is inscribed in a given circle, are together equal to the remaining angles of the figure; any angle whatever being assumed as the first.*

Let $ABCDEF$ be any polygon, of an even number of sides, inscribed in the given circle ACE: Then A being assumed as the first angle, the $\angle A + \angle C + \angle E +$, &c. $= \angle B + \angle D + \angle F +$, &c.



First, let the inscribed figure have six sides, and join B, E.

Then, since $BAFE$ is a quadrilateral figure inscribed in a circle, therefore (E. xxii. 3.) the

$$\angle BAF + \angle FEB = \angle EFA + \angle EBA:$$

Also, the $\angle BCD + \angle BED = \angle EDC + \angle EBC$.

Wherefore, equals being added to equals, it will be manifest, that the $\angle BAF + \angle BCD + \angle FED = \angle CBA + \angle EDC + \angle AFE$:

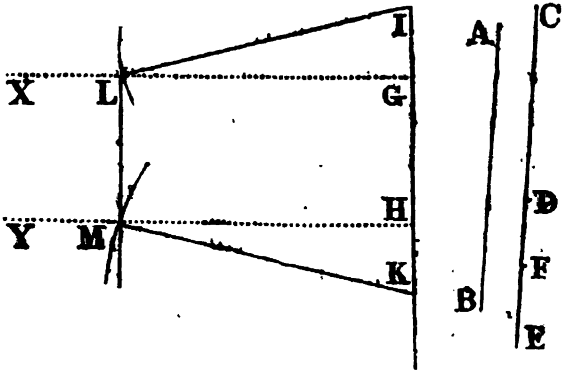
i.e. the $\angle A + \angle C + \angle E = \angle B + \angle D + \angle F$.

And, in a similar manner, the proposition may be demonstrated, when the figure inscribed in the given circle has eight, ten, twelve, or any other even number of sides.

PROP. XXXII.

42. PROBLEM. *To make a trapezium, about which a circle may be described, having its four sides respectively equal to four given straight lines, two of which are equal to each other, and any three together greater than the fourth; the two equal sides of the trapezium, also, being opposite to each other.*

Let *AB*, *CD*, *DE* be three straight lines: It



is required to make a trapezium having two of its opposite sides each of them equal to *AB*, and its two other sides equal to *CD* and *CE*, each to each, about which a circle may be described.

Take $\overline{GH} = \overline{CD}$; and *CD* and *CE* being placed in the same straight line, so that *DE* shall be their difference, bisect (E. x. 1.) DE in F; produce GH, both ways, and make GI and HK each of them equal to DF or FE; $\therefore IK=CE$: From the points G, H draw (E. xi. 1.) $\overline{GX}$ and $\overline{HY}$ perpendicular to IK; from I and K, as centres, at distances equal to AB, describe two circles, cutting GX and HY in L and M, respectively; and join I, L and K, M; therefore (E. xv. Def. 1.) $IL=AB$ and $KM=AB$; join L, M.

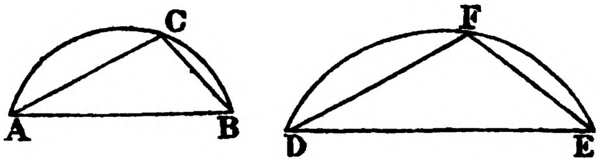
And, because (*constr.*) $LI=MK$, and $IG=KH$, and that the $\angle$s IGL, KHM, are right angles, (Supp. lxxiv. 1.) $GL=HM$; and, since the angles at G and H are right angles, GL is (E. xxviii. 1.) parallel to HM; therefore (E. xxxiii. 1.) LM is parallel and equal to EH; but (*constr.*) $GH=CD$; therefore $LM=CD$.

Again, since $GLMH$ is a parallelogram, the $\angle GLH=\angle GHM$ (E. xxxiv. 1.) which (*constr.*) is a right angle; also, since the two sides IL, LG, of the $\triangle LGI$, are equal to the two sides KM, MH of the $\triangle MHK$, and the base IG is equal to the base KH; therefore (E. viii. 1.) the $\angle ILG=\angle KMH$; but (*constr.* and E. xxxii. 1.) the $\angle HKM+\angle KMH$ = a right angle; therefore the $\angle HKM$, or IKM, $+ ILG$ = a right angle; to each of these add the right $\angle GLM$; therefore the $\angle IKM+\angle ILG+\angle GLM$ = two right angles; that is, the $\angle IKM+\angle ILM$ = two right angles; therefore (Supp. xxviii. 3.) a circle may be described about the trapezium $ILMK$, which, as hath been shewn, has two equal sides LI, MK, each of them equal to AB, has its side LM equal to CD, and its remaining side IK equal to CG.

PROP. XXXIII.

43. **PROBLEM.** *Upon a given finite straight line to describe a segment of a circle, which shall be similar to a given segment of another circle.*

Let ACB be a segment of a circle, and DE a finite straight line: It is required to describe on DE a segment of a circle, similar to the segment ACB.

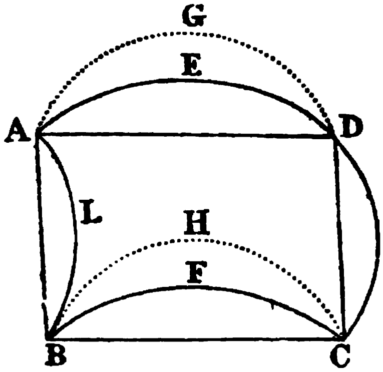


In $\overset{\frown}{ACB}$ take any point C, and join A, C, and B, C: At the point D in $\overline{DE}$ make (E. xxiii. 1.) the $\angle EDF = \angle BAC$; and, at the point E, also, make the $\angle DEF = \angle ABC$; therefore (Supp. xxvi. 1.) the $\angle DFE = \angle ACB$: About the $\triangle DFE$ describe (Supp. v. 1. *Cor.*) the circle DFE; therefore (E. xi. Def. 3. and E. xxi. 3.) the segment DFE is similar to the segment ACB.

PROP. XXXIV.

44. **THEOREM.** *If upon two opposite sides of an oblong, two similar segments of circles be described, the one of them lying wholly within the oblong, and the other wholly without it, the figure contained by the two remaining sides of the oblong and the two circular arches, shall be equal to the oblong.*

Upon the two opposite sides *AD*, *BC*, of the oblong *ABCD*, let there be described two similar segments of circles *AED*, *BFC*: the one, namely *BFC*, lying wholly within the oblong, and the other lying wholly without it: The figure contained by $\overline{BA}$, $\overset{\frown}{AED}$, $\overline{DC}$ and $\overset{\frown}{CFB}$ is equal to the oblong *ABCD*.



For (*hyp.* and E. xxxiv. 1.) $AD = BC$; therefore (*hyp.* and E. xxiv. 3.) the segment $AED =$ the segment *BFC*; to each of these equals add the figure *ADCFB*, and it is manifest that the figure *AEDCFB* is equal to the oblong *ABCD*.

45. Cor. 1. An indefinite number of such mixtilineal figures may be found (Supp. iii. 1. *Cor.* 3. and Supp. xxxiii. 3.) equal to one another, and each of them equal to any given oblong.

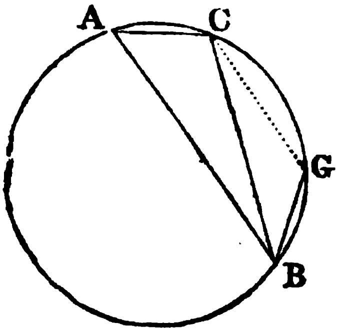
46. Cor. 2. If upon *AB* and *DC*, the two remaining sides of the oblong, there be, likewise, described two similar segments of circles *ALB*, *DKC*, it is evident that the figure *ALBCDA* is equal to the figure *ABFCDEA*; and that the figure $ALBFCKDE = ABCD$, *ALB* being supposed not to meet *BFC* again within *ABCD*.

PROP. XXXV.

47. THEOREM. *The arches of a circle that are intercepted between two parallel chords are equal to one another.*

Let *AB* and *CG* be two parallel chords of the circle *ACGB*: Then is $\overset{\frown}{AC} = \overset{\frown}{BG}$.

For join *C, B*: And, because (*hyp.*) *CG* is parallel to *AB*; therefore (E. xxix. 1.) the $\angle GCB$

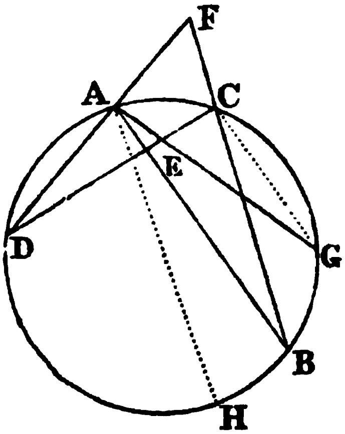


$= \angle CBA$; therefore (E. xxvi. 3.) $\overset{\frown}{AC} = \overset{\frown}{BG}$.

PROP. XXXVI.

48. THEOREM. *If two chords of a given circle intersect each other, the angle of their inclination is equal to the half of the angle at the centre standing upon the aggregate, or the difference, of the arches intercepted between them, accordingly as they meet within, or without the circle.*

First, let the two chords *AB, CD*, of the circle *ACBD*, cut one another in *E*, within the circle: The $\angle DEB$ is equal to the half of an angle at the centre, standing upon a circumference equal to $\overset{\frown}{AC}+\overset{\frown}{DB}$.



For through C draw (E. xxxi. 1.) CG parallel to AB; therefore $\overset{\frown}{BG}=\overset{\frown}{AC}$, and $\overset{\frown}{DBG}=\overset{\frown}{AC}+\overset{\frown}{DB}$; but (*constr.* and E. xxix. 1.) the $\angle DEB=\angle DCG$, which (E. xx. 3.) is the half of an angle at the centre, standing upon $\overset{\frown}{DBG}$.

Secondly, let the two chords DA and BC, meet when produced, without the circle, in F: If, then, AH be drawn parallel to CB, it may be shewn, in a similar manner, that the $\angle DFB$ is equal to the half of an angle at the centre standing on $\overset{\frown}{DH}$, which is the difference between $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$.

PROP. XXXVII.

49. THEOREM. *In equal circles the greater angle stands upon the greater circumference; whether the angles compared be at the centres or the circumferences.*

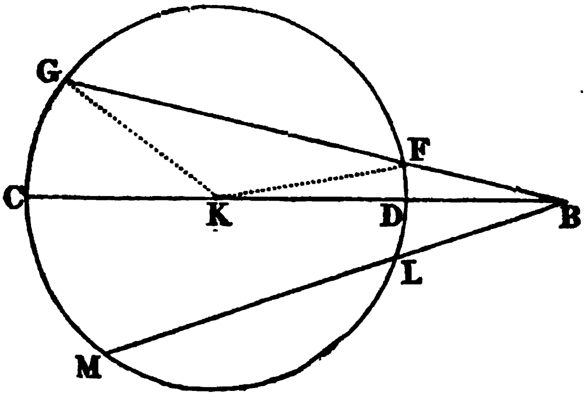
For whether the angles be at the centres, or the circumferences, if, from the greater, an angle

(E. xxiii. 1.) be cut off equal to the less, the circumference on which it stands, will evidently be part of the circumference on which the greater angle stands, and will (E. xxvi. 3.) be equal to that on which the less angle stands; the which circumference is, therefore, less than the other.

PROP. XXXVIII.

50. THEOREM. *If from any given point, without a circle, there be drawn two straight lines cutting the circle, then of the circumferences which they intercept, that which is the nearer to the given point is less than the other.*

From the given point *B* without the circle *FDCG*,



let there be drawn $\overline{BFG}$, $\overline{BDC}$, cutting the circumference in the points *F, G,* and *D, C,* respectively: Then is $\widehat{FD} < \widehat{GC}$.

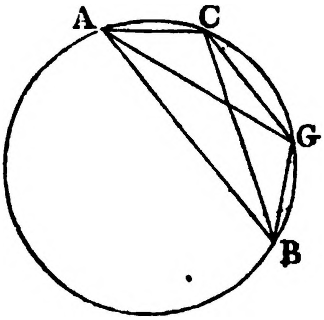
First, let one of the straight lines drawn from *B*, as *BC*, pass through the centre *K* of the circle: Join *K, F* and *K, G*; then (E. xvi. 1.) the exterior $\angle GKC$, of the $\triangle GKB$, is $> \angle KGF$; but (E. xv. Def. 1. and E. v. 1.) the $\angle KGF = \angle KFG$; therefore the $\angle GKC > \angle KFG$; and (E. xvi. 1.) the $\angle KFG > \angle FKB$ or FKD; much more, then, is the $\angle GKC > \angle FKD$; therefore (Supp. xxxvii. 3.) $\overset{\frown}{CG} > \overset{\frown}{FD}$; *i. e.* $\overset{\frown}{FD} < \overset{\frown}{GC}$.

But if BLM do not pass through the centre, find (E. i. 3.) the centre K; join B, K; and produce it to meet the circumference in C: Then it may be shewn, as before, that $\overset{\frown}{FD} < \overset{\frown}{GC}$, and that $\overset{\frown}{DL} < \overset{\frown}{CM}$; $\therefore \overset{\frown}{FDL} < \overset{\frown}{GCM}$.

PROP. XXXIX.

51. THEOREM. *The straight lines joining the extremities of the chords of two equal arches of the same circle, toward the same parts, are parallel to each other.*

Let $\overline{AC}$, $\overline{BG}$ be the chords of two equal arches



$\overset{\frown}{AC}$, $\overset{\frown}{BG}$, of the circle $ABGC$; and let A, B, and C, G be joined: Then $\overline{CG}$ is parallel to $\overline{AB}$.

For join C, B; and since (*hyp.*) $\overset{\frown}{AC} = \overset{\frown}{BG}$; therefore (E. xxvii. 3.) the $\angle ABC = \angle BCG$; therefore (E. xxvii. 1.) $\overline{CG}$ is parallel to $\overline{AB}$.

PROP. XL.

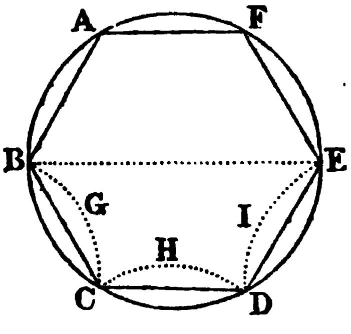
52. THEOREM. *In equal circles, the greater of two circumferences subtends the greater angle, whether the angles compared be at the centres or the circumferences.*

For if not, the angle standing on the greater circumference is equal to the other angle or less than it; but it cannot be equal; for then (E. xxvi. 3.) the two circumferences would be equal, which is contrary to the hypothesis: Neither can it be less, for, then, (Supp. xxxvii. 3.) the greater circumference would be less than the other, which is absurd. Wherefore, the greater of two circumferences subtends the greater angle, whether the two angles be at the centres, or circumferences.

PROP. XLI.

53. PROBLEM. *If any equilateral rectilineal figure, of an* even *number of sides, be inscribed in a given circle, to find a curvilineal figure that is equal to it, and that is bounded by arches of circles, each of which circles is equal to the given circle.*

Let *ABCDEF* be an equilateral rectilineal figure, of an *even* number of sides, inscribed in the given



circle *ACE*; It is required to find a curvilineal figure equal to it, and bounded by arches of circles that are equal to the given circle *ACE*.

On half the number of sides of the inscribed figure, taken contiguous to one another, as *BC*, *CD*, *DE*, describe (Supp. xxxviii. 3.) segments of circles, *BGC*, *CHD*, *DIE*, each similar to the segment cut off from the given circle by each of the sides: The curvilineal figure contained by $\overset{\frown}{BGC}$, $\overset{\frown}{CHD}$, $\overset{\frown}{DIE}$, $\overset{\frown}{EF}$, and $\overset{\frown}{FA}$, and $\overset{\frown}{AB}$, is equal to the inscribed polygon *ABCDEF*.

For, since (*hyp.*) the figure is equilateral, (E. xxviii. 3.) $\overset{\frown}{AB} = \overset{\frown}{BC}$; $\therefore$ $\overset{\frown}{AFEDCB} = \overset{\frown}{BAFEDC}$; therefore (E. xxvii. 3.) the angle in the segment cut off by $\overline{AB}$ is equal to the angle in the segment cut off by $\overline{BC}$; therefore these two segments (E. xi. Def. 3.) are similar, and (*hyp.* and E. xxiv. 3.) equal to one another.

And, in the same manner, may all the segments, cut off by the equal sides of the inscribed figure, be shewn to be similar and equal to one another, and to the segments *BGC*, *CHD*, *DIE*.

But the figure contained by $\overline{BA}$, $\overline{AF}$, $\overline{FE}$, $\overset{\frown}{EID}$, $\overset{\frown}{DHC}$ and $\overset{\frown}{CGB}$, together with the segments *BGC*, *CHD*, *DIE*, makes up the equilateral rectilineal figure *ABCDEF*; and that same figure, together with the equal segments cut off by *BA*, *AF*, and *FE*, makes up the curvilineal figure contained by $\overset{\frown}{BGC}$, $\overset{\frown}{CHD}$, $\overset{\frown}{DIE}$, $\overset{\frown}{EF}$, $\overset{\frown}{FA}$ and $\overset{\frown}{AB}$; the which figure is, therefore, equal to the inscribed rectilineal figure *ABCDEF**.

PROP. XLII.

54. THEOREM. *In equal circles, the greater chord subtends the greater circumference.*

For (*hyp.* and E. xv. Def. 1. and E. xxv. 1.) the angle subtended, at the centre, by the greater chord is greater than the angle subtended, at the centre, by the less: therefore (Supp. xxxvii. 3.) the circumference subtended by the greater chord is greater than the circumference subtended by the less.

PROP. XLIII.

55. THEOREM. *If from a given point within a circle, which is not the centre, straight lines be drawn to the circumference, making with each other equal angles, the two, which are nearer to the diameter passing through the given point, shall cut off a greater circumference than the two, which are more remote.*

From *A*, a point within the circle *BDC*, let there be drawn to the circumference any number

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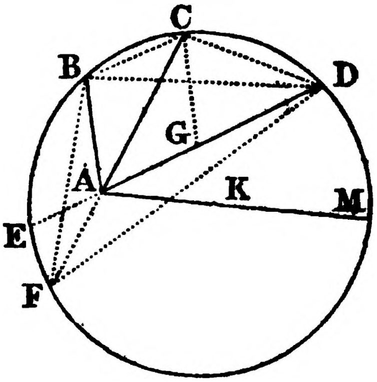
* It is easy to shew, by the help, chiefly, of Supp. vii. 3., that when the equilateral figure inscribed in the circle, is a square, the circumferences of the similar segments, described in the course of the demonstration, touch one another in the common extremity of the two contiguous sides; and that when the inscribed polygon has any greater number of sides, as six, eight, &c., the circumferences of any two of the segments meeting one another in the common extremity of two contiguous sides, do not meet again within the circle.



of straight lines *AB*, *AC*, *AD*, &c. containing equal $\angle$s *BAC*, *CAD*, &c.; and let $\overline{AKM}$ be drawn, from *A*, through the centre *K*: Then is $\overset{\frown}{CD} > \overset{\frown}{CB}$.

For produce *DA* and *CA*, to meet the circumference again, in *E* and *F*; and join *B*, *F* and *D*, *F*: Then, since *AD* is nearer than *AB* is to *AM*, therefore (E. vii. 3.) $AD > AB$; from *AD* cut off $AG = AB$, and join *C*, *G* and *C*, *B*, and *C*, *D* and *D*, *B*: And, because *CA* is common to the two $\triangle$s *CBA*, *CGA*, and $AB = AG$, and that (*hyp.*) the $\angle CAB = \angle CAG$; $\therefore$ $CG = CB$. Again, because (E. xxxii. 1.) the $\angle CGD = \angle GCA + \angle CAG = \angle ACB + \angle BAC$, and that (E. xvi. 1.) the $\angle BAC > \angle BFC$, therefore the $\angle CGD > \angle FCB + \angle BFC$: But (E. xxi. 3.) the $\angle FCB = \angle FDB$, and the $\angle BFC = \angle BDC$; therefore the $\angle CGD > \angle FDB + \angle BDC$; *i. e.* the $\angle CGD > \angle FDC$, much more then is the $\angle CGD > \angle EDC$ or $\angle GDC$; therefore (E. xix. 1.) $CD > CG$; but it has been shewn that $CG = CB$; $\therefore$ $\overline{CD} > \overline{CB}$; therefore (Supp. xlii. 3.) $\overset{\frown}{CD} > \overset{\frown}{CC}$.

PROP. XLIV.

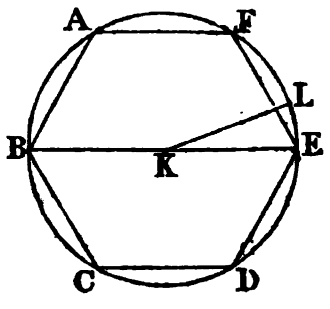
56. **THEOREM.** *In equal circles, the greater circumference has the greater chord.*

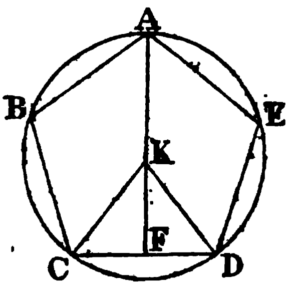
For (Supp. xl. 3.) the greater circumference subtends the greater angle at the centre; therefore (E. xv. Def. 1. and E. xxiv. 1.) it has the greater chord.

PROP. XLV.

57. **THEOREM.** *The straight line joining any of the angular points of an equilateral polygon inscribed in a circle and the centre, passes through the opposite angular point, or else bisects the opposite side at right angles, accordingly as the figure has an* even, *or an* odd *number of sides.*

First, let the equilateral polygon *ABCDEF* inscribed in the circle *ACE,* of which the centre is *K,* have an even number of sides, and let *B*, *K* be joined, *B* being any one of the angular points of the inscribed figure: Then *BK* passes through the opposite angular point *E*.





For, if it be possible, let *BK* cut the circumference in any other point L; therefore (E. xxviii. 3.) $\overset{\frown}{BAL}$ is the half of the whole circumference; also, since the polygon (*hyp.*) is equilateral, the arches $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, $\overset{\frown}{DE}$, $\overset{\frown}{EF}$, $\overset{\frown}{FA}$ are (E. xxviii. 3.) equal to one another; therefore $\overset{\frown}{BAE}$ is the half of the whole circumference; $\therefore$ $\overset{\frown}{BAL} = \overset{\frown}{BAE}$, the less to the greater, which is absurd: therefore BK, produced, passes through E.

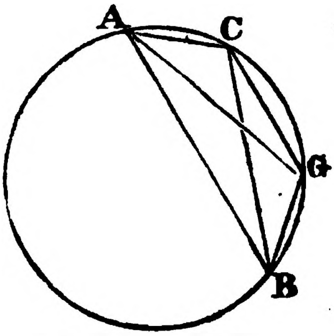
But, secondly, let the equilateral polygon $ABCDE$, inscribed in the circle ACE, have an odd number of sides, and let any of the angular points, as A, and the centre K, be joined: Then $\overline{AK}$, produced, bisects CD, at right angles in the point F.

For join K, C and K, D: And, because (*hyp.*) the sides of the inscribed polygon are equal, the circumferences which they subtend are (E. xxviii. 3.) equal; since, therefore, the polygon has an odd number of sides, it is manifest that the circumference ABC is equal to the circumference AED; therefore (E. xxvii. 3.) the $\angle AKC = \angle AKD$; therefore (E. xiii. 1.) the $\angle CKF = \angle DKF$; and (E. xv. Def. 1.) $CK = DK$, and KF is common to the two $\triangle$s KFC, KFD; therefore (E. iv. 1.) $CF = FD$, and the $\angle KFC = \angle KFD$; so that $\overline{AKF}$ bisects, at right angles, the side CD, which is opposite to the $\angle BAE$.

PROP. XLVI.

58. THEOREM. *The two straight lines in a circle, which join the extremities of two parallel chords, are equal to each other.*

Let $\overline{AB}$, $\overline{CG}$ be two parallel chords of the circle $ABGC$, and let their extremities be joined, toward



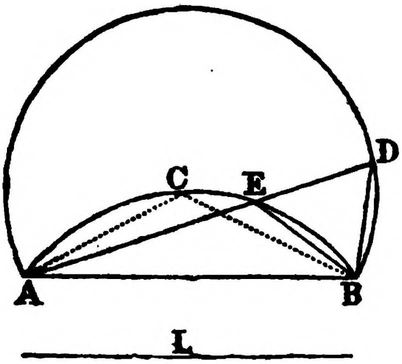
the same parts, by $\overline{CA}$ and $\overline{GB}$, and towards opposite parts by $\overline{CB}$ and $\overline{GA}$: Then $CA = GB$, and $CB = GA$.

For, since CG is parallel to AB, $\overset{\frown}{CA} = \overset{\frown}{GB}$ (Supp. xxxv. 3.); therefore (E. xxix. 3.) $CA = GB$: Again, since it has been shewn that $\overset{\frown}{AC} = \overset{\frown}{BG}$, to each of these add $\overset{\frown}{CG}$; $\therefore$ $\overset{\frown}{ACG} = \overset{\frown}{BGC}$; therefore (E. xxix. 3.) $GA = CB$.

PROP. XLVII.

59. PROBLEM. *To divide a given circular arch into two parts, so that the aggregate of their chords may be equal to a given straight line, greater than the chord of the whole arch, but not greater than the double of the chord of half the arch.*

Let $\overset{\frown}{ACB}$ be a circular arch, of which $\overline{AB}$ is the chord; and let L be a finite straight line, greater than AB, but not greater than twice the chord of the half of $\overset{\frown}{ACB}$: It is required to divide



$\overset{\frown}{ACB}$ into two parts such that the aggregate of their chords shall be equal to L.

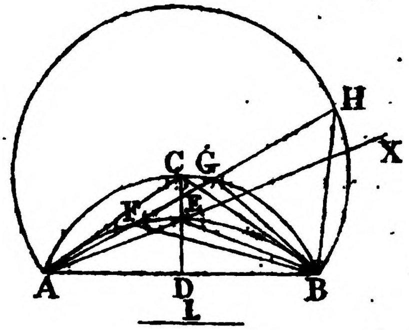
Bisect (E. xxx. 3.) $\overset{\frown}{ACB}$ in C, and join C, A, and C, B; therefore (E. xxix. 3.) $\overline{CA} = \overline{CB}$; from the centre C, at the distance CA, or CB, describe the circle ADB, which will, therefore, pass through both A and B: From the centre A, at a distance equal to L, describe a circle, cutting the circle ADB in D; draw $\overline{AD}$, which is, therefore, equal to L; let $\overline{AD}$ cut $\overset{\frown}{ACB}$ in E, and join E, B: Then $\overline{AE}+\overline{EB} = L$.

For (E. xx. 3.) the $\angle ACB$ is the double of the $\angle ADB$; and (E. xxi. 3.) the $\angle ACB = \angle AEB$; therefore the $\angle AEB$ is the double of the $\angle ADB$; but (E. xxxii. 1.) the $\angle AEB = \angle EDB + \angle EBD$; therefore the $\angle EDB + \angle EBD$ is equal to the double of the $\angle EDB$; from these equals take the $\angle EDB$, and there remains the $\angle EBD = \angle EDB$; therefore (E. vi. 1.) $ED = EB$; $\therefore$ $AE + EB = AE + ED$ or AD; but (constr.) $AD = L$; $\therefore$ $AE + EB = L$.

PROP. XLVIII.

60. PROBLEM. *To divide a given circular arch into two parts; so that the excess of the chord of the one above the chord of the other, may be equal to a given straight line, less than the chord of the whole arch.*

Let *ACB* be a circular arch, of which the chord is *AB*: It is required to divide $\overset{\frown}{ACB}$ into two parts, such that the excess of the chord of the one above the chord of the other shall be equal to a given finite straight line *L*, that is less than *AB*.



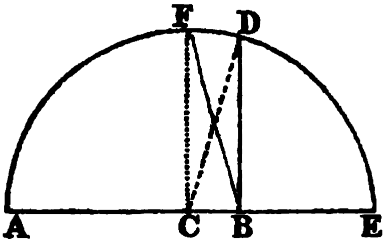
Bisect *AB* (E. x. 1.) in *D*; draw $\overline{DC}$ (E. xi. 1.) perpendicular to *DB*; join *A*, *C*; bisect (E. ix. 1.) the $\angle CAB$ by $\overline{AX}$; let *AX* meet *CD* in *E*; and join *B*, *E*; about the $\triangle AEB$ describe (Supp. v. 1. Cor.) the circle *AEB*; from the centre *A*, at a distance $= L$, describe a circle cutting *AEB* in *F*; draw $\overline{AF}$, which is, therefore, equal to *L*, and produce *AF* to meet $\overset{\frown}{ACB}$ in *G*; join *G*, *B*: Then is the excess of *AG* above *GB* equal to *L*.

For join C, B; and (*constr.* and E. iv. 1.) $CA = CB$ and the $\angle ACD = \angle BCD$; also $EA = EB$, and the $\angle ACB$ is the double of the $\angle ACD$; Again, from the centre C at the distance CA, describe the circle AHB, which, because $CA = CB$, passes through B; produce $\overline{AG}$ to meet $\overset{\frown}{AHB}$ in H; join H, B, and F, B: And since (E. xxi. 3.) the $\angle AFB = \angle AEB$, therefore (E. xiii. 1.) the $\angle BFH = \angle BEX$; but (E. xxxii. 1. and E. v. 1.) the $\angle BEX$ is the double of the $\angle BAE$; therefore the $\angle BFH$ is the double of the $\angle BAE$, and is therefore (*constr.*) equal to the $\angle CAB$ or CAD; also (E. xx. 3. and *constr.*) the $\angle ACB$ is the double of the $\angle AHB$; and it is also, as hath been shewn, the double of the $\angle ACD$; therefore the $\angle ACD = \angle AHB$ or FHB; and it has been proved that the $\angle HFB$, of the $\triangle FBH$, is equal to the $\angle CAD$ of the $\triangle CDA$; therefore (Supp. xxvi. 1.) the $\angle HBF = \angle CDA$, and is, therefore, a right angle; but (*demonstr.* of Supp. xlvii. 3.) $GH = GB$; therefore (Supp. xxix. 1. *Cor.* 3.) $GF = GB$; $\therefore AG - GB = AF$: but (*constr.*) $AF = L$; $\therefore AG - GB = L$.

PROP. XLIX.

61. THEOREM. *If from any point, in the diameter of a semi-circle, there be drawn two straight lines to the circumference, one to the bisection of the circumference, the other at right angles to the diameter, the squares upon these two lines are, together, the double of the square upon the semi-diameter.*

Let B be any point in the diameter AE of the semi-circle *ADE*; let *F* be the bisection of the circumference *ADE*; and let *C* be the bisection



of the diameter: If *B*, *F* be joined, and *BD* be drawn perpendicular to *AE*, then $\overline{BF}^2 + \overline{BD}^2 = 2\overline{AC}^2$.

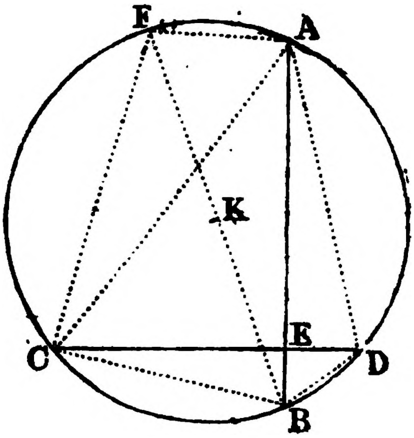
For join *C*, *F* and *C*, *D*: And since (*hyp.*) $\widehat{AF} = \widehat{EF}$; $\therefore$ (E. xxvii. 3.) the $\angle ACF = \angle ECF$; and they are adjacent angles; therefore (E. x. Def. 1.) the $\angle FCE$ is a right angle; therefore (E. xlvii. 1.) $BF^2 = \overline{CF}^2 + \overline{CB}^2$; to each of these add BD^2; therefore $BF^2 + BD^2 = CF^2 + CB^2 + BD^2$; but (*hyp.* and E. xlvii. 1.) $CB^2 + BD^2 = CD^2$; $\therefore$ $BF^2 + BD^2 = CF^2 + CD^2$ or (E. xv. Def. 1.) $2\,AC^2$.

PROP. L.

62. THEOREM. *If the chords of two arches of any the same circle cut each other at right angles, the squares of the four segments of the chords, are, together, equal to the square of the diameter.*

Let the two chords *AB*, *CD* of the circle *ACD*, cut each other at right angles, in *E*: The squares of the segments of the chords are, together, equal to the square of the diameter of the circle.

For find (E. i. 3.) the centre K, and from either extremity of either of the chords, as B, draw through



K the diameter BKF; join B, C and C, F, and F, A and A, D. And since (*constr.*) $FADB$ is a semi-circle, the $\angle FAB$ is (E. xxxi. 3.) a right angle, as is, also, (*hyp.*) the $\angle AEC$; therefore (E. xxviii. 1.) FA is parallel to CD; therefore (Supp. xliv. 3.) $\overline{FC} = \overline{AD}$. Again, because FCB is a semi-circle, the $\angle BCF$ (E. xxxi. 3.) is a right angle; therefore (E. xlvii. 1.) $\overline{FB}^2 = \overline{FC}^2 + \overline{CB}^2$; but FC has been shewn to be equal to AD; $\therefore$ $FB^2 = AD^2 + CB^2$; that is, (*hyp.* and E. xlvii. 1.)

$$FB^2 = \overline{AE}^2 + \overline{ED}^2 + \overline{CE}^2 + \overline{EB}^2.$$

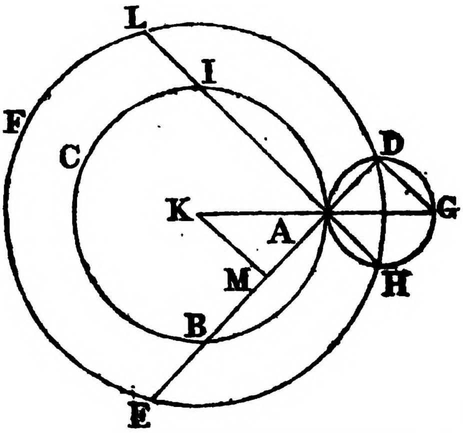
63. Cor. If the diagonals of a quadrilateral rectilineal figure, inscribed in a circle, cut each other at right angles, the aggregate of the squares of the sides is the double of the square of the diameter of the circle.

For let the diagonals AB and CD of the quadrilateral figure $ACBD$, inscribed in the circle $ACBD$, cut one another at right angles in E: Then it is evident from E. xlvii. 1, that $\overline{AC}^2+\overline{DB}^2$ is equal to the squares of the segments of the diagonals, that is, (Supp. I. 3.) to the square of the diameter of the circle: Likewise AD^2+CB^2 may, in the same manner, be shewn to be equal to the square of the diameter; $\therefore\ AC^2+CB^2+\overline{BD}^2+\overline{DA}^2 =$ twice the square of the diameter of the circle.

PROP. LI.

64. PROBLEM. *To draw a straight line, cutting two concentric circles, the diameter of the greater of which does not exceed the double of the diameter of the less, so that the part of it which lies within the greater circle may be the double of the part which lies within the less.*

Let ABC, DEF be two circles, having a common



centre K, the diameter of DEF being not greater than the double of the diameter of ABC: It is required to draw a straight line cutting ABC, DEF, so that the part of it within DEF shall be the double of the part of it within ABC.

Take any semi-diameter as KA, of the circle

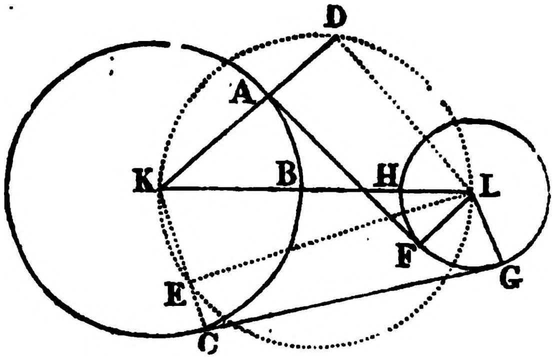
ABC, and produce it to G, so that $AG=AK$; upon AG as a diameter describe the circle $DAHG$ cutting DEF in D and H; join D, A, and H, A; and produce $\overline{DA}$ and $\overline{HA}$ to meet the circumferences again in B, E, and I, L, respectively: Then $DE=2AB$; and $HL=2AI$.

For join D, G and draw (E. xii. 1.) KM perpendicular to AB: And since (*constr.*) ADG is a semicircle, the $\angle ADG$ (E. xxxi. 3.) is a right angle, as is (*constr.*) the $\angle KMA$; also (E. xv. 1.) the $\angle KAM = \angle DAG$; and (*constr.*) the side KA, of the $\triangle KMA$, is equal to the side AG, of the $\triangle ADG$; therefore (E. xxvi. 1.) $AD=AM$; $\therefore MD=2MA$, but (*constr.* and E. iii. 3.) $DE=2MD$ and $AB=2MA$; $\therefore DE=2AB$: And, in the same manner, HL may be shewn to be the double of AI.

PROP. LII.

65. PROBLEM. *To draw a straight line which shall touch two given circles.*

Let ABC, HFG, be two circles: It is required



to draw a straight line which shall touch both the circles ABC, HFG.

First let the two circles be unequal. Find (E. i. 3.) the centres K and L, of the two circles ABC, HFG; join K, L; upon KL as a diameter describe the circle DKC; from K as a centre at a distance $=KB+LH$, the aggregate of the semi-diameters of the two circles, describe a circle cutting $\overset{\frown}{KDL}$ in D, and draw $\overline{KD}$, cutting the circumference of ABC in A; $\therefore$ $KD=KB+LH$; in like manner, by the help of E. iii. 1, place, in the semi-circle KCL, $KE=KB\sim LH$, and let KE produced meet the circumference of ABC in C; from L draw (E. xxxi. 1.) LF parallel to DK, and LG parallel to KC; lastly, join A, F, and C, G: Then will $\overline{AF}$ and $\overline{CG}$, each of them, touch both the circles ABC, HFG.

For join D, L, and E, L: And since (*constr.*) $KE=KC\sim LG$, it is manifest that $EC=LG$; and (*constr.*) EC is parallel to LG; therefore (E. xxxiii. 1.) CG is parallel to LE; but, since KEL is a semi-circle, the $\angle KEL$ is (E. xxxi. 3.) a right angle; therefore (E. xxix. 1.) the $\angle$s KCG, CGL are right angles; therefore (E. xvi. 3. *Cor.*) CG touches both the circles ABC, HFG.

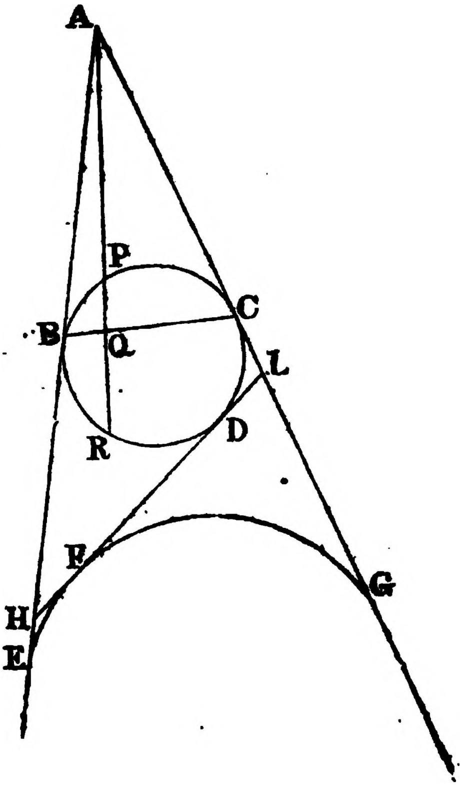
And, in the same manner, it may be shewn, that AF touches both the circles ABC, HFG.

Secondly, if the two given circles be equal to one another, a straight line may be drawn which shall touch them on contrary sides, in the same manner as when they are unequal: And, it is manifest (E. xxxiii. 1. E. xxix. 1. and E. xvi. 3. *Cor.*) that if a semi-diameter in each circle be drawn perpendicular to the straight line joining the two centres, the straight line, which joins the extremities of these two semi-diameters, will touch both the circles on the same side.

PROP. LIII.

66. THEOREM. *If two straight lines, which touch two given circles, the one touching both the circles on the one side of them, the other on the other, be cut by a third tangent, which touches the two circles on contrary sides of them, then, of the segments into which the two first tangents are thus divided, those which are alternate are equal to one another.*

Let *ABE*, *ACG* touch the two given circles *BCD*, *EFG*, *ABE* on the one side of them, and *ACG* on the other; and let *HLFD* be drawn (Supp.



lii. 3.) touching the two circles, on contrary sides of them: Of the segments into which $\overline{HL}$ divides $\overline{BE}$ and $\overline{CG}$, $\overline{BH}=\overline{LG}$, and $\overline{EH}=\overline{CL}$.

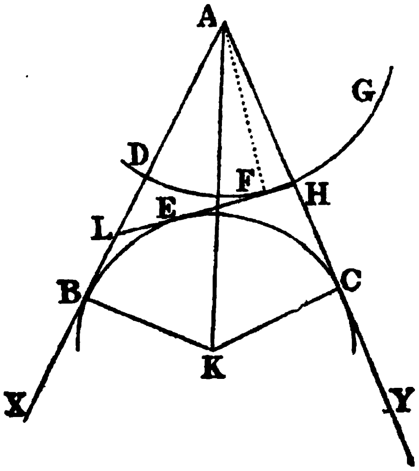
If the two circles be equal to one another, it is manifest, from the latter part of the demonstration of Supp. lii. 3., that BE and CG will be opposite sides of a parallelogram, and that, therefore, (E. xxxiv. 1.) $BE=CG$: And, if BE be not parallel to CG, but meets it, both the lines being produced, in A, then, since (Supp. xix. 3. *Cor.* 1.) $\overline{AE}=\overline{AG}$ and $\overline{AB}=\overline{AC}$; ∴ $BE=CG$, as in the former case.

Again, (Supp. xix. 3. *Cor.* 1.) $\overline{HB}+\overline{LC}=\overline{HD}+\overline{LD}$; or $\overline{HL}$; and $\overline{HL}=\overline{HF}+FL=\overline{HE}+\overline{LG}$; ∴ $HB+LC=HE+LG$; and, as hath been shewn, $BE=CG$; that is, $HB+HE=LC+LG$; if, therefore, these two equals be added to the equals $HB+LC$ and $HE+LG$, it is evident that $2HB+HE+LC=2LG+HE+LC$; take away from both $HE+LC$, and there remains $2HB=2LG$; ∴ $HB=LG$: And it has been proved that $BE=CG$; if, therefore, from these equals there be taken the equals HB and LG, there will remain $EH=CL$.

PROP. LIV.

67. PROBLEM. *The perimeter, the vertical angle, and the altitude of a triangle being given, to construct the triangle.*

Let XAY be a rectilineal angle: It is required to describe a triangle, which shall have XAY for its vertical angle, which shall have a given perimeter, and the perpendicular drawn from *A*



to the opposite side, equal, also, to a given straight line.

From *AX* and *AY* cut off *AB* and *AC*, each of them equal to the half of the given perimeter; from *B* and *C* draw (E. xi. 1.) *BK* and *CK* perpendicular to *AB* and *AC*, respectively, and join *A*, *K*; therefore (*constr.* and Supp. lxxiii. 1.) *KB*=*KC*; from the centre *K*, at the distance *KB*, describe the circle *BEC*, which (*constr.* and E. xvi. 3. *Cor.*) will touch *AB* and *AC* in the points *B* and *C*; from *AX* cut off *AD* equal to the given perpendicular, and from the centre *A* at the distance *AD* describe the circle *DFG*.

Lastly, draw (Supp. lii. 3.) the straight line *LH* touching the circle *BEC* in *E*, and the circle *DFG* in *F*: Then is *ALH* the triangle which was to be described.

For it has the ∠*XAY* for its vertical angle, and if *A*, *F* be joined, since $\overline{LH}$ touches the circle *DFG* in F, the $\angle$s AFL, AFH, are (E. xviii. 3.) right angles; and (E. xv. Def. 1.) $AF=AD$ which was made equal to the given perpendicular: Also (*constr.* and Supp. xix. 3. *Cor.* 1.) $LE=LB$, and $HE=HC$; $\therefore LE+HE$, *i. e.* $LH=LB+HC$; to these equals add $AL+AH$; $\therefore AL+LH+HA=AL+LB+AH+HC$; but $AL+LB=AB$, and $AH+HC=AC$; $\therefore AL+LH+HA=AB+AC$; and AB and AC were made each of them equal to the half of the given perimeter; therefore the $\triangle ALH$ has its perimeter equal to the given perimeter; it has the given angle for its vertical angle, and, as hath been shewn, it has its perpendicular AF equal to the given altitude.

PROP. LV.

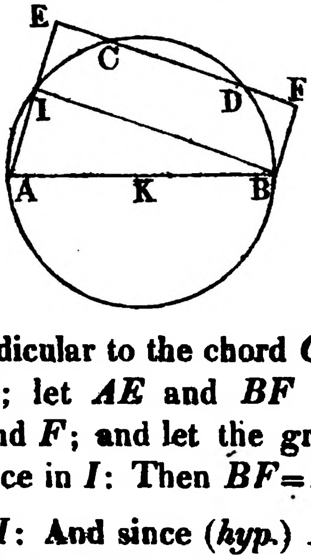
68. THEOREM. *If the point, in which two straight lines that are perpendicular to each other meet, be applied to the circumference of a circle so that the straight lines themselves cut the circumference, the centre of the circle is in the bisection of the straight line joining those two intersections.*

For, the straight line, joining the intersections of the circumference and of the two straight lines which (*hyp.*) meet at some point of the circumference, and contain a right angle, cuts off a semi-circle: If not, let it, if it be possible, cut off a segment greater than a semi-circle; therefore (E. xxxi. 3.) the angle in that segment is less than a right angle, which is contrary to the supposition: Neither can it cut off a segment less than a semi-circle; for, then the angle in that segment would be greater than a right angle, which is, also, contrary to the supposition; therefore the straight line joining the intersections cuts off a semi-circle, and therefore passes through the centre of the circle, which point is therefore in the bisection of that line.

PROP. LVI.

69. THEOREM. *If from the extremities of any diameter, of a given circle, perpendiculars be drawn to any chord of the circle, that is not parallel to the diameter, the less perpendicular shall be equal to the segment of the greater contained between the circumference and the chord.*

From the extremities, *A* and *B*, of the diameter *AB* of the circle *ABDC*, let there be drawn *AE*



and *BF* perpendicular to the chord *CD*, which is not parallel to *AB*; let *AE* and *BF* meet *CD*, produced, in *E* and *F*; and let the greater $\perp$ *AE* cut the circumference in *I*: Then $BF = IE$.

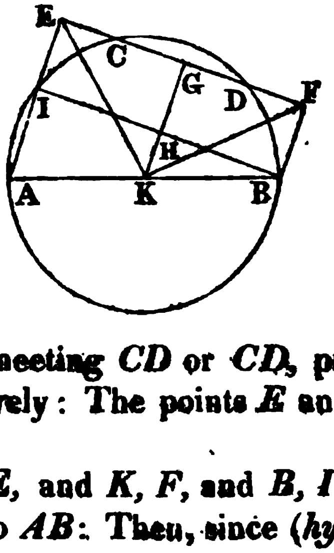
For join *B*, *I*: And since (*hyp.*) $\overset{\frown}{AICB}$ is a semicircle, therefore (E. xxxi. 3.) the $\angle AIB$, is a right angle; and (*hyp.*) the $\angle$s *IEF*, *EFD* are, also, right angles; therefore (E. xxviii. 1.) the figure *IEFB* is a parallelogram; therefore (E. xxxiv. 1.) $BF = IE$.

And in the same manner may the proposition be proved, when the chord *CD* cuts *AB* within the circle.

Prop. LVII.

70. Theorem. *If from the extremities of any diameter, of a given circle, perpendiculars be drawn to any chord of the circle, they shall meet the chord, or the chord produced, in two points which are equidistant from the centre.*

From the extremities, *A*, *B*, of the diameter *AB*, of the circle *ABCD*, of which *K* is the centre, let there be drawn *AE*, and *BF*, perpendicular to the chord *CD*, and meeting *CD* or *CD*, produced in *E* and *F*, respectively: The points *E* and *F* are equidistant from *K*.



For, join *K*, *E*, and *K*, *F*, and *B*, *I*; and first let *EF* be parallel to *AB*: Then, since (*hyp.*) the angles *AEF*, *EFB* are right angles, therefore (E. xxviii. 1.) *EA* is, also, parallel to *FB*, and *AEFB* is a parallelogram; therefore (E. xxxiv. 1.) *EA*=*FB*, and the ∠*EAB*= ∠*FBA*, each of them being a right angle; also the side *KA*, of the △*EAK*, is equal to the side *KB*, of the △*FKB*; therefore (E. iv. 1.) *EK*=*FK*.

But, if EF be not parallel to AB, join B, I, and through K draw (E. xxxi. 1.) KHG parallel to AE or BF: And since (*hyp.*) $AICD$ is a semi-circle, therefore (E. xxxi. 3.) the $\angle AIB$ is a right angle, as is, also, (*hyp.*) the $\angle AEF$; therefore (E. xxviii. 1.) EF is parallel to IB; and (*constr.*) the figures IG, HF, are parallelograms; therefore (E. xxxiv. 1.) $EG=IH$, and $GF=HB$; but since KG (*constr.*) is parallel to AE, and that the angles AIB, AEF are right angles; therefore (E. xxix. 1.) the $\angle$s IHK, EGH are right angles; and because KH, drawn from the centre K, cuts IB at right angles, therefore (E. iii. 3.) $IH=HB$; $\therefore$ $EG=GF$; and KG is common to the two triangles, KGE, KGF, and the $\angle$s KGE, KGF, as hath been shewn, are right angles; therefore (E. iv. 1.) $EK=FK$.

And in the same manner may the proposition be proved, when the chord CD cuts the diameter AB within the circle[*].

PROP. LVIII.

71. THEOREM. *If upon either radius, bounding a quadrantal circular arch, as a diameter, a semi-circle be described, any chord of the semi-circle, drawn from the centre of the quadrant, shall be equal to the perpendicular distance of the point, in which the chord produced meets the quadrantal arch, from the other radius.*

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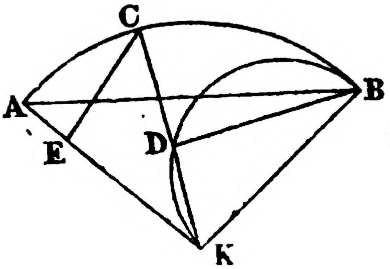
[*] It is manifest from the demonstration, and from E. iii. 3., that the segment EB is equal to the segment FD.

Let *BDK* be a semi-circle, having for its diameter



KB, one of the semi-diameters which bound the quadrantal circular arch $\overset{\frown}{ACB}$, and from the point *C*, in which any chord *KD*, of the semi-circle, meets, when produced, $\overset{\frown}{ACB}$, let *CE* be drawn perpendicular to *KA* the other terminating semi-diameter of $\overset{\frown}{ACB}$: Then *KD* = *CE*.

For, since (*hyp.*) *KDB* is a semi-circle; therefore (E. xxxi. 3.) the ∠ *BDK* is a right angle; as is, also, (*hyp.*) the ∠ *CEK*: Again, since (*hyp.*) *ACB* is a quadrant of the circumference of its circle; therefore (E. xxvii. 3. and E. xv. 1. *Cor.* 2.) the ∠ *AKB* is a right angle; therefore the ∠ *EKC* + ∠ *DKB* = a right angle; also, since the ∠ *CEK* is a right angle, therefore (E xxxii. 1.) the ∠ *EKC* + ∠ *KCE* = a right angle; therefore the ∠ *EKC* + ∠ *DKB* = ∠ *EKC* + ∠ *KCE*; therefore the ∠ *DKB* = ∠ *KCE*; and the side *KB*, of the △ *KDB*, is (E. xv. Def. 1.) equal to the side *KC*, of the △ *CEK*; therefore (E. xxvi. 1.) *KD* = *CE*.

PROP. LIX.

72. THEOREM. *If the angle contained by two straight lines, one of which cuts a circle and the*

other meets it, be equal to the angle in the alternate segment of the circle, the straight line which meets, shall touch the circle.

For if the straight line which, in this case, meets the circle, does not touch it, from the point in which it meets the circle, draw (E. xvii. 3.) a straight line touching the circle: Then (*hyp.* and E. xxxii. 3.) it is manifest that the greater of two angles is equal to the less; which is absurd.

Prop. LX.

73. Theorem. *A straight line touching a circular arch in the bisection of that arch, is parallel to its chord.*

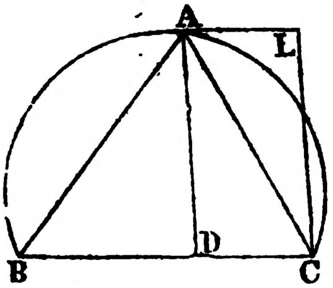
For the angles, which each half of the arch subtends at the opposite extremity of the chord, are (E. xxvii. 3.) equal to one another; and (E. xxxii. 3.) they are also equal to the angles which the straight lines, joining the bisection of the arch and the extremities of the chord, make with the straight line that touches the arch at its bisection; therefore (E. xxvii. 1.) the tangent, at that point of bisection, is parallel to the chord.

Prop. LXI.

74. Problem. *The base, the vertical angle, and the altitude of a triangle being given, to construct the triangle.*

Let *BC* be the given base of a triangle, of which the vertical angle, and the altitude are also given: It is required to construct the triangle.

Upon *BC* describe (E. xxxiii. 3.) a segment of a circle *BAC*, capable of containing an angle equal to



the given vertical angle; from *C* draw (E. xi. 1.) $\overline{CL}$ perpendicular to *BC*, and make it equal to the given altitude of the triangle; through *L* draw $\overline{LA}$ parallel to $\overline{BC}$, and let *LA* meet $\overset{\frown}{BAC}$ in *A*; join *B*, *A* and *C*, *A*: Then is *ABC* the triangle which was to be constructed.

For draw (E. xxxi. 1.) *AD* parallel to *LC*; therefore, the figure *ADCL* is a parallelogram, and (E. xxxiv. 1.) $AD = LC$: And because *AD* is parallel to *LC*, and (*constr.*) the $\angle LCD$ is a right angle; therefore (E. xxix. 1.) the $\angle ADC$ is a right angle; *i. e.* $\overline{AD}$ is perpendicular to *BC*, and it has been shewn to be equal to *LC*, which (*constr.*) is equal to the given perpendicular. Also (*constr.*) the $\angle BAC$ is equal to the given vertical angle; $\therefore$ *ABC* is the triangle which was to be constructed[*].

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[*] If the straight line *LA* drawn from the extremity of *CL*, which is made equal to the given perpendicular, fall without the segment *BAC*, the problem is manifestly impossible: If *LA* touch the circle *BAC*, the problem has only one solution; but if *LA* cut the segment *BAC*, the problem admits of two solutions.

PROP. LXII.

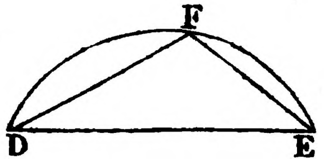
75. **PROBLEM.** *To find a point in a given straight line, from which if straight lines be drawn to two given points, on the same side of the given line, they shall contain an angle equal to a given rectilineal angle.*

Upon the straight line joining the two given points, describe (E. xxxiii. 3.) a segment of a circle, capable of containing an angle equal to the given rectilineal angle, and the point in which it meets, or cuts, the given straight line, is evidently the point which was to be found: And if the circumference of the segment, so described, cut the given straight line, it is manifest that the problem admits of two solutions: But if the circumference of the segment neither touch nor cut the given line, the problem is impossible.

PROP. LXIII.

76. **PROBLEM.** *The vertical angle, the base, and the aggregate of the three sides of a triangle being given, to construct the triangle.*

Let *DE* be the given base: It is required to describe on *DE* a triangle, which shall have its two remaining sides equal together, to a given finite straight line, and the angle contained by them equal to a given rectilineal angle.

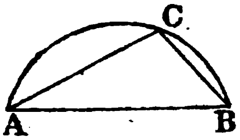


Upon *DE* describe (E. xxxiii. 3.) a segment of a circle *DFE*, capable of containing an angle equal to the given rectilineal angle; divide (Supp. xlvii. 3.) $\overset{\frown}{DFE}$, in *F*, so that the aggregate of the chords of $\overset{\frown}{DF}$, $\overset{\frown}{EF}$ shall be equal to the aggregate of the two remaining sides of the triangle; and join *D*, *F* and *E*, *F*; Then it is manifest that *DFE* is the triangle which was to be constructed.

PROP. LXIV.

77. PROBLEM. *The vertical angle, the base, and the excess of the greater of the two remaining sides, of a triangle, above the less, being given, to construct the triangle.*

Let *AB* be the given base: It is required to describe on *AB* a triangle, which shall have the difference of its two remaining sides equal to a given finite straight line, and the angle contained by them equal to a given rectilineal angle.

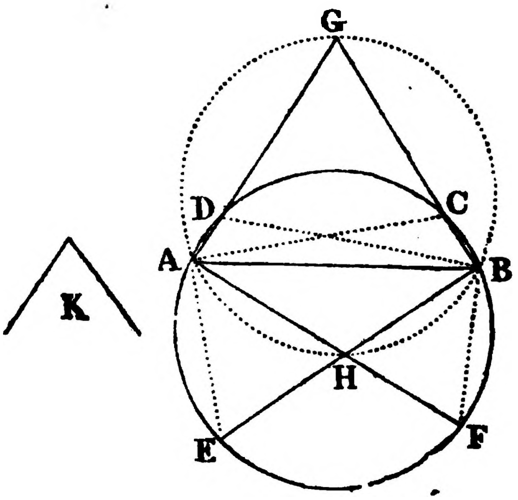


Upon *AB* describe (E. xxxiii. 3.) a segment of a circle capable of containing an angle equal to the given rectilineal angle; divide (Supp. xlviii. 3.) $\overset{\frown}{ACB}$ in *C*, so that the difference of the chords of $\overset{\frown}{AC}$ and $\overset{\frown}{BC}$ may be equal to the excess of the greater of the two remaining sides of the triangle above the less; and join *A*, *C* and *B*, *C*: Then it is evident that *ACB* is the triangle which was to be constructed.

PROP. LXV.

78. PROBLEM. *From two given points, in the circumference of a circle, to draw two equal chords of that circle, which, produced if necessary, shall make with one another an angle equal to a given rectilineal angle.*

Let *A* and *B* be two points in the circumference of the circle *AEFB*; and let *K* be a rectilineal angle: It is required to draw, from *A* and *B*, two equal chords of the circle *AEFB*, which make with one another an angle equal to the angle *K*.



Join *A*, *B*; upon $\overline{AB}$ describe (E. xxxiii. 3.) a segment of a circle *AGB* capable of containing an angle equal to the $\angle K$, and complete the circle *AGBH*; bisect (E. xxx. 3.) $\overset{\frown}{AGB}$ in *G*, and $\overset{\frown}{AHB}$ in *H*; draw $\overline{AG}$ and $\overline{BG}$, cutting $\overset{\frown}{ADB}$ in *D* and *C*;

also draw $\overline{AH}$ and $\overline{BH}$, and produce them to meet $\overset{\frown}{AEB}$ in F and E.

It is manifest, from the construction, that the $\angle AGB$, which the two chords, AD, BC, make with one another when produced, $= \angle K$; also, since (E. xxii. 3.) the $\angle AGB + \angle AHB =$ two right angles, and that (E. xiii. 1.) the $\angle AHE + \angle AHB =$ two right angles; therefore the $\angle AHE = \angle AGB$; but (*constr.*) the $\angle AGB = \angle K$; therefore the $\angle AHE$, which the two chords AF, BE, make with one another, is equal to the $\angle K$.

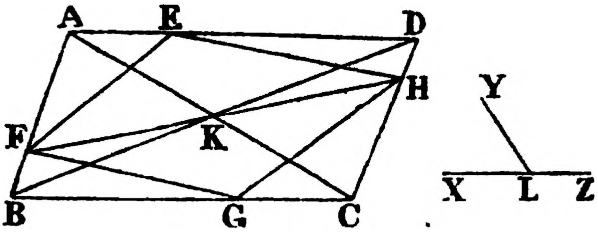
Again, join A, C, and B, D; and since (*constr.*) $\overset{\frown}{AG} = \overset{\frown}{BG}$, therefore (E. xxxvii. 3.) the $\angle GAB = \angle GBA$; therefore (E. xxvi. 3.) $\overset{\frown}{DCB} = \overset{\frown}{CDA}$; take away the common part $\overset{\frown}{DC}$, and $\overset{\frown}{AD} = \overset{\frown}{BC}$; therefore (E. xxix. 3.) $\overline{AD} = \overline{BC}$.

Lastly if A, E, and B, F, be joined, it may be shewn, in the same manner, that the $\angle HAB = \angle HBA$, and that (E. xxvi. 3.) $\overset{\frown}{AE} = \overset{\frown}{BF}$; add to these equals $\overset{\frown}{EF}$, and $\overset{\frown}{AEF} = \overset{\frown}{BFE}$; $\therefore$ (E. xxix. 3.) $\overline{AF} = \overline{BE}$.

PROP. LXVI.

79. **PROBLEM.** *In a given parallelogram to inscribe a parallelogram which shall have one of its angles equal to a given angle, and posited in a given point of one of the sides of the given parallelogram.*

Let F be a point in the side AB of the $\square ABCD$; and YLX a given rectilineal angle: It is required to inscribe, in the ▱ *ABCD*, a parallelogram



which shall have one of its angles posited in *F*, and equal to the ∠ *YLX*.

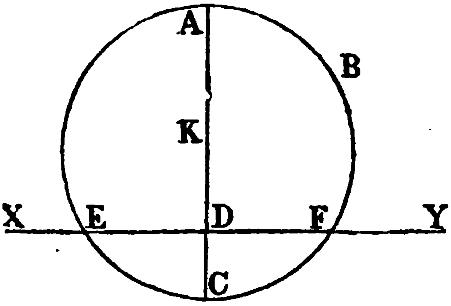
Join *B*, *D*; bisect $\overline{BD}$ (E. x. 1.) in *K*; draw $\overline{FK}$ and produce it to meet *DC* in *H*; therefore (*constr.* E. xv. 1. E. xxix. 1. and E. xxvi. 1.) $DH = BF$; produce *XL* to *Z*; upon *FH* describe (E. xxxiii. 3.) a segment of a circle capable of containing an angle equal to the ∠ *YLZ*, and let its circumference cut *AD* in *E*; if therefore *F*, *E* and *H*, *E* be joined, it is plain that the ∠ $FEH =$ ∠ *YLZ*; from *CB* cut off $CG = AE$; join *F*, *G* and *H*, *G*: Then is *FEHG* the figure which was to be described.

For (*constr.* and Supp. xliii. 1.) *EFGH* is a parallelogram; therefore (E. xxix. 1.) the ∠ *HEF* + ∠ *EFG* = two right angles; also (E. iii. 1.) the ∠ *YLZ* + ∠ *YLX* = two right angles; and it has been shewn, that the ∠ *HEF* = ∠ *YLZ*; therefore the ∠ *EFG* = ∠ *YLX*; and it is posited in the given point *F*. Therefore, &c.

PROP. LXVII.

80. PROBLEM. *To produce a given straight line so that the rectangle, under the given straight line, and the part of it produced, shall be equal to a given square.*

Let AD be a finite straight line: It is required



to produce AD, so that the rectangle contained by AD and the part produced may be equal to a given square.

Through D draw any straight line XY, and from DX and DY cut off (E. iii. 1.) DE and DF each of them equal to the side of the given square; describe (Supp. v. 1. *Cor.*) a circle which shall pass through the three points A, E, and F; and produce AD to meet the circumference in C: Then (E. xxxv. 3.) it is manifest that the rectangle $\overline{AD}\times\overline{DC}$ $=\overline{ED}\times\overline{DF}$ or ED^2; but ED was made equal to the side of the given square; ∴ AD has been produced to C, so that $AD\times DC$ is equal to the given square.

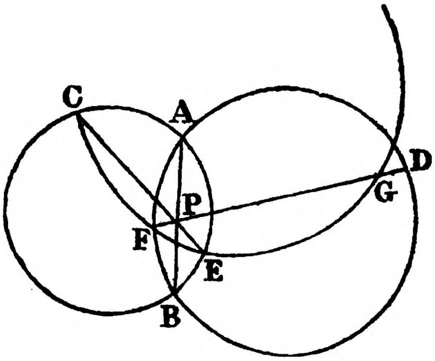
81. Cor. By a similar method, it is manifest, a given straight line may be produced, so that the rectangle contained by the straight line, and the part produced shall be equal to a given rectangle: That is, if three straight lines be given, a fourth may be found so that the rectangle, contained by it and one of the three given straight lines, shall be equal to the rectangle contained by the remaining two.

PROP. LXVIII.

82. THEOREM. *If through any point in the common chord of two circles, which intersect one another, there be drawn any two other chords, one in each circle, their four extremities shall all lie in the circumference of a circle.*

Let *P* be any point in $\overline{AB}$, which is a common chord of the two circles *ABC, ABD*; and through *P* let there be drawn a chord *CPE*, of the circle *ABC*, and *FPD* a chord of the circle *ABD*; the four points *C, F, E, D,* lie in the circumference of a circle.

For describe (Supp. v. 1. *Cor.*) a circle *CFE*, which shall pass through the three points *C, F* and *E*; it shall, also, pass through *D*: If not let it cut *FD* in some other point as *G*.

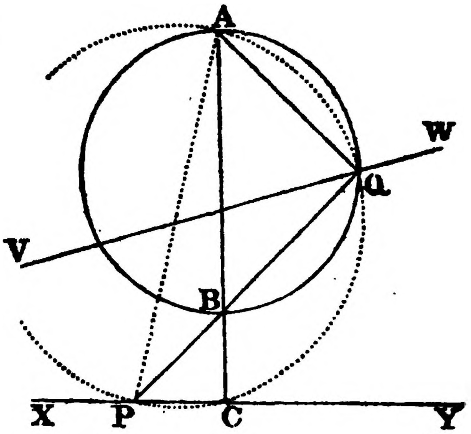


Then (E. xxxv. 3.) $\overline{CP} \times \overline{PE} = \overline{AP} \times \overline{PB}$; and $AP \times PB = \overline{FP} \times \overline{PD}$; ∴ $CP \times PE = FP \times PD$; but (E. xxxv. 3.) $CP \times PE = FP \times \overline{PG}$; ∴ $FP \times PD = FP \times PG$; *i. e.* the greater rectangle is equal to the less, which is absurd; therefore the circle which passes through *C, F, E* cannot pass otherwise than through *D*.

PROP. LXIX.

83. **THEOREM.** *If through the given extremity of any diameter of a circle straight lines be drawn to meet an indefinite straight line without the circle, which is perpendicular to the diameter produced, the rectangles contained by the segments of these lines lying between the given point, the point in which each of them cuts the circumference again, and the indefinite line, shall be equal to each other.*

Through the extremity *B* of the diameter *AB*, of the circle *AQB*, let there be drawn any number of straight lines, terminated one way by the circumference, and the other way by the indefinite straight line *XY*, which meets *AB*, produced, at right angles in *C*: The rectangles contained by the segments into which the lines so drawn are divided by the point *B*, shall be equal to one another.



For, let $\overline{PBQ}$ be any of the lines so drawn through *B*; join *A, P* and *A, Q*: And because (*hyp.*) *AQB* is a semi-circle; therefore (E. xxxi. 3.) the $\angle AQP$

is a right angle, as is, also, (*hyp.*) the $\angle ACP$; therefore (Supp. xxix. 1. *Cor.* 2.) a circle described upon AP as a diameter, will pass through Q and C; therefore (E. xxxv. 3.) $\overline{PB} \times \overline{BQ} = \overline{AB} \times \overline{BC}$; and, in the same manner, it may be shewn that the rectangle contained by the segments of any other straight line, so drawn through B, is equal to $AB \times BC$, and, therefore, equal also to $PB \times BQ$. All such rectangles are, therefore, equal to one another.

84. COR. Hence, through a given point, (B) between an indefinite straight line (XY) and a line of any kind (VW), in the same plane with it, a straight line may be drawn to meet the two given lines, so that the rectangle, contained by the segments into which it is divided by the given point, shall be equal to a given square.

For draw (E. xii. 1.) BC perpendicular to XY, and produce CB (Supp. lxvii. 3.) to A, so that $\overline{CB} \times \overline{BA}$ may be equal to the given square; upon AB, as a diameter, describe the circle ABQ*, cutting VW in Q; lastly, draw QB, and produce it to meet XY in P: Then since (Supp. lxix. 3.) $PB \times BQ = CB \times BA$, and that (*constr.*) $CB \times BA$ is equal to the given square; $\therefore$ $PB \times BQ$ is, also, equal to the given square.

PROP. LXX.

85. PROBLEM. *From the obtuse angle of an obtuse-angled triangle, to draw a straight line to the*

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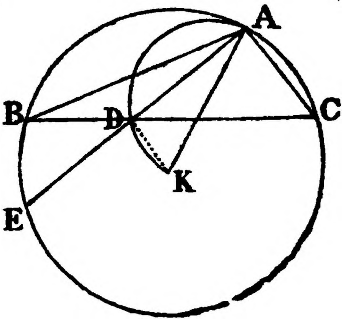
* If the circumference of the circle ABQ do not cut VW, the problem admits not of a solution.

base, the square of which shall be equal to the rectangle contained by the segments, into which it divides the base.

Let *BAC* be an obtuse-angled triangle, obtuse-angled at *A*: It is required to draw from *A* to *BC* a straight line, the square of which shall be equal to the rectangle of the segments into which it divides *BC*.



About *BAC* describe (Supp. v. 1. *Cor.*) a circle *ABEC*, and take its centre *K*; join *K*, *A*; upon *KA*, as a diameter, describe the circle *ADK* cutting *BC* in *D*; join *A*, *D*: Then $\overline{AD}^2 = \overline{BD} \times \overline{DC}$.

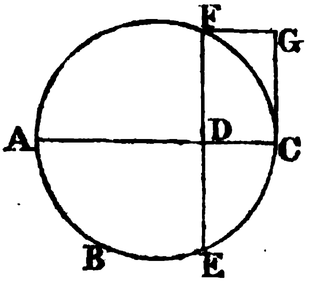
For produce *AD* to meet the circumference in *E*, and join *K*, *D*: And since (*constr.*) *ADK* is a semicircle; therefore (E. xxxi. 3.) the ∠ *ADK* is a right angle; therefore (E. iii. 3.) $AD = \overline{DE}$; but (E. xxxv. 3.) $BD \times DC = AD \times DE$; $\therefore BD \times DC = AD^2$.

86. Cor. A segment of a circle being given, that is less than a semi-circle, the method of drawing, from any point of its circumference, a chord of the circle, that shall be bisected by the chord of the segment, is shewn in the solution of the above problem.

Prop. LXXI.

87. Problem. *To make a rectangle which shall be equal to a given square, and shall have its two adjacent sides, together, equal to a given straight line; the side of the given square being less than the half of the given straight line.*

Let AC be a straight line: It is required to make a rectangle, which shall be equal to a given square, and which shall have its two adjacent sides, together, equal to $\overline{AC}$.



Upon AC, as a diameter, describe the circle ABC; from C draw (E. xi. 1.) $\overline{CG}$ perpendicular to AC, and make CG equal to the side of the given square; through G draw, (E. xxxi. 1.) $\overline{GF}$ parallel to AC, and through F draw $\overline{FDE}$ parallel to CG: The rectangle $\overline{AD} \times \overline{DC}$ is equal to the given square.

For (*constr.*) $DCGF$ is a parallelogram; therefore (E. xxxiv. 1.) $DF = CG$, and therefore (*constr.*) DF = the side of the given square: Again, because (*constr.*) DF is parallel to CG, and the ∠ ACG is a right angle; therefore (E. xxix. 1.) the ∠ CDF is, also, a right angle; and (*constr.*) $\overline{ADC}$ is the diameter of the circle ABC; therefore (E. iii. 3.) DF

$=DE$; but (E. xxxv. 3.) $AD \times DC = \overline{DF} \times \overline{DE}$; *i. e.*, since $DF = DE$, $AD \times DC = DF^2$; $\therefore AD \times DC$ is equal to the given square, and AD together with DC make up the given straight line AC.

88. Cor. 1. If the side of the given square be greater than the half of the given straight line, the problem admits of no solution.

89. Cor. 2. In the same manner, the greater side of a given oblong may be divided into two parts, so that the rectangle contained by them shall be equal to the given oblong, a square having first (E. xiv. 2.) been found that is equal to the oblong: But, in this case, the half the greater side of the oblong must exceed the double of the lesser side.

90. Cor. 3. In the same manner, also, a straight line may be divided into two parts, so that the rectangle contained by them, shall be equal to a given rectangle; if the side of a square which is equal to the given rectangle, do not exceed the half of the given straight line.

91. Cor. 4. If the measure of the surface of an oblong be given, and if its perimeter be also given, the rectangle itself may hence be constructed.

Prop. LXXII.

92. Theorem. *If from a given point without a circle, two equal straight lines be drawn to the convex circumference, one of which touches the circle, the other shall also touch it.*

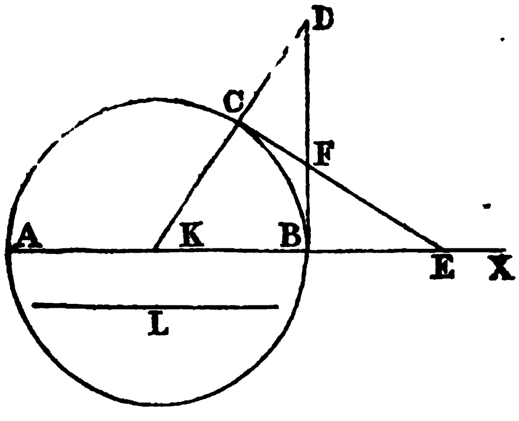
For, if not, draw (E. xvii. 3.) from the given point a straight line to touch the circle, and (Supp. xix. 3.

Cor. 1.) it will be equal to the other tangent; and thus more than two equal straight lines can be drawn from a point, without a circle, to the circumference; which (E. viii. 3.) is absurd; therefore the straight line which is drawn from the same point, without the circle, as the tangent, and which is equal to the tangent, itself also touches the circle.

PROP. LXXIII.

93. PROBLEM. *To produce a given straight line, so that the rectangle contained by the whole line thus produced, and the part of it produced, shall be equal to a given square.*

Let *AB* be a straight line, and *L* the side of a square: It is required to produce *AB* so that the rectangle contained under the whole line produced, and the part of it produced, may be equal to the square of *L*.



Bisect (E. x. 1.) *AB* in *K*, and upon *AB*, as a diameter, describe the circle *ABC*; from *B* draw (E. xi. 1.) $\overline{BD}$ perpendicular to $\overline{AB}$, and make *BD* $= L$; join *K*, *D*, and let $\overline{KD}$ cut the circumference in C; from C draw $\overline{CE}$ perpendicular to $\overline{KC}$, and let CE meet $\overline{AB}$ produced in E. Then since (*constr.* and E. xvi. 3. *Cor.*) CE touches the circle, and EBA cuts it; therefore (E. xxxvi. 3.) $\overline{AE} \times \overline{EB} = \overline{EC}^2$; but, since the ∠s KBD, KCE are right angles, and the angle at K is common to the two △s KBD, KCE, and that (E. xv. Def. 1.) the side $KD =$ the side KC; therefore (E. xxvi. 1.) $EC = BD$; but (*constr.*) $BD = L$; and it has been shewn that $AE \times EB = EC^2$; ∴ $AE \times EB =$ the square of L.

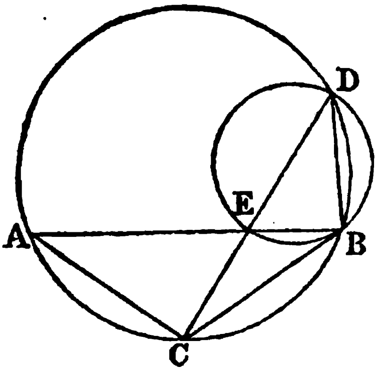
94. **Cor.** By the help of this proposition and (E. xiv. 2.), a given straight line may be produced, so that the rectangle contained by the whole line thus produced, and the part of it produced, shall be equal to a given rectilineal figure.

Prop. LXXIV.

95. **Theorem.** *If, from the bisection of any given arch of a circle, a straight line be drawn cutting the chord of that arch, or the chord produced, and the circumference also of the circle, the rectangle contained by the two parts of the straight line so drawn, the one lying between the point of bisection and the circumference, the other between the point of bisection and the chord, shall be equal to the square of the chord of half the arch.*

Let AB be the chord and let C be the bisection, of the arch $\overset{\frown}{ACB}$ of the circle $ADBC$; and, first, let any straight line CD be drawn cutting the chord in E, and then meeting the circumference of the circle in D; also let there be drawn $\overline{CB}$, the chord of $\overset{\frown}{CB}$, the half of $\overset{\frown}{ACB}$: Then $\overline{DC} \times \overline{CE} = CB^2$.

For join *C*, *A* and *B*, *D*, and about the $\triangle DBE$



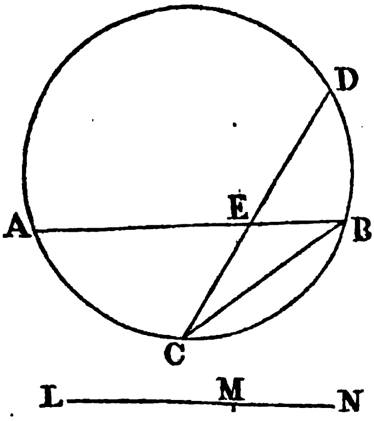
describe (Supp. v. 1. *Cor.*) the circle *DEB*: And because (*hyp.*) $\overset{\frown}{AC} = \overset{\frown}{CB}$; therefore (E. xxvii. 3.) the $\angle ABC = \angle CAB$; and (E. xxi. 3.) the $\angle CAB = \angle CDB$; therefore the $\angle ABC = \angle BDE$; therefore (Supp. lix. 3.) the straight line *CB* touches the circle *DEB* in *B*; therefore (E. xxxvi. 3.) $\overline{DC} \times \overline{CE} = \overline{CB}^2$.

Prop. LXXV.

96. Problem. *From the bisection of a given arch of a circle, to draw a straight line, such that the part of it intercepted between the chord, or the chord produced, of the given arch and the circumference, shall be equal to a given straight line.*

Let $\overset{\frown}{ACB}$ be an arch of the circle *ADBC*; let $\overline{AB}$ be its chord, and *C* its bisection, and let *LM* be a given straight line: It is required to draw from *C* a straight line such that the part of it between *AB*, and the circumference *ADB* shall be equal to *LM*.

Join *C*, *B*; and produce (Supp. lxxiii. 3.) *LM* to *N*, so that $\overline{LN} \times \overline{NM} = \overline{CB}^2$; from *C* as a centre, at a distance equal to LN, describe a circle cutting



$\overset{\frown}{ADB}$ in D; join C, D, and let $\overline{CD}$ cut $\overline{AB}$ in E: Then is $\overline{ED}=\overline{LM}$.

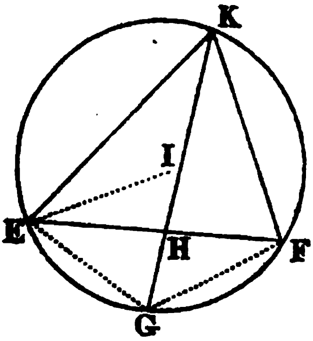
For (Supp. lxxiv. 3.) $\overline{DC} \times \overline{CE} = \overline{CB}^2$; and (*constr.*) $\overline{LN}\times\overline{NM}=CB^2$; $\therefore DC\times CE=LN\times NM$; but (*constr.*) $DC=LN$; $\therefore CE=NM$; and therefore $ED=LM$.

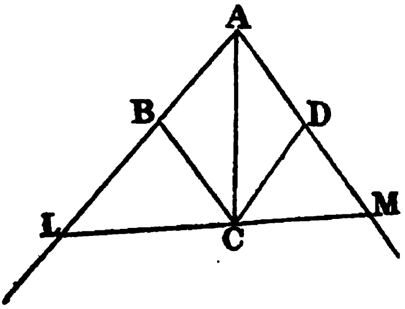
PROP. LXXVI.

97. PROBLEM. *Through any given angle of a given equilateral four-sided figure, to draw a straight line terminated by the sides produced, containing the angle opposite to the given angle, which shall be equal to a given straight line.*

Let $ABCD$ be an equilateral four-sided figure, and EF a straight line: Through any of the angular points of $ABCD$, as C, it is required to draw a straight line, terminated by AB and AD produced, which shall be equal to $\overline{EF}$.

Join A, C; upon EF describe (E. xxxiii. 3.) a segment of a circle *EKF*, capable of containing an





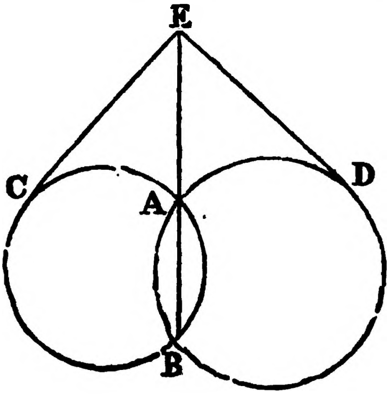
angle equal to the $\angle BAD$ of the rhombus, and complete the circle; bisect (E. xxx. 3.) $\overset{\frown}{EGF}$ in *G*; from *G* draw (Supp. lxxv. 3.) *GHK* so that $HK = AC$; join *E*, *K*; in *AB* produced, take (E. iii. 1.) $AL = KE$; join *L*, *C*, and produce *LC* to meet *AD* produced in *M*: Then $\overline{LCM} = \overline{EF}$.

For join *K*, *F*, and *G*, *E*, and *G*, *F*: And because (E. xxxii. Def. 1.) *BA*, *AC* are equal to *DA*, *AC*, each to each, and that the base *BC*, of the $\triangle ABC$, is equal to the base *DC*, of the $\triangle ADC$, therefore (E. viii. 1.) the $\angle BAC = \angle DAC$, and the $\angle BAC$ is, therefore, the half of the $\angle BAD$: Again, because (*constr.*) $\overset{\frown}{EG} = \overset{\frown}{FG}$; therefore (E. xxvii. 3.) the $\angle EKG = \angle FKG$, and the $\angle EKG$ is, therefore, the half of the $\angle EKF$, which (*constr.*) is equal to the $\angle BAD$; therefore the $\angle EKH = \angle LAC$; and (*constr.*) the two sides *EK*, *KH* of the $\triangle EKH$, are equal to the two sides *LA*, *AC*, of the $\triangle LAC$; therefore (E. iv. 1.) $LC = EH$, and the $\angle ACL = \angle KHE$; therefore (E. xiii. 1.) the $\angle KHF = \angle ACM$; also, as hath been shewn, the $\angle CAM = \angle HKF$, and the side *KH* (*constr.*) of the $\triangle KHF$, is equal to the side AC of the $\triangle$ ACM; therefore (E. xxvi. 1.) $CM=HF$; and it has been proved that $LC=EH$; $\therefore LC+CM=EH+HF$; that is, $LM=EF$.

PROP. LXXVII.

98. THEOREM. *If two circles cut each other, and from any point, in the straight line produced, which joins their intersections, two tangents be drawn, one to each circle, they shall be equal to one another.*

Let the two circles ACB, ADB, cut one another



in the points A and B, and from any point E in AB, produced, let there be drawn $\overline{EC}$ and $\overline{ED}$ touching the circles ACB, ADB, in the points C and D respectively: $EC=ED$.

For (E. xxxvi. 3.) $EC^2=\overline{BE}\times\overline{EA}$; also $ED^2=BE\times EA$; $\therefore EC^2=ED^2$; and $\therefore EC=ED$.

99. COR. The straight line AB which passes through the intersections of two circles ACB, ADB, that cut one another, bisects the straight line HL*, which touches both the circles.

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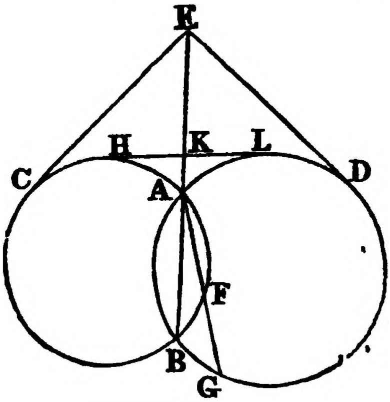
* See the diagram to the next Proposition.

PROP. LXXVIII.

100. **THEOREM.** *If two circles cut each other, and if two tangents drawn, one to each circle, from any point without them, be equal, the straight line, joining the intersections of the circles, shall, if it be produced, pass through the common extremity of the equal tangents.*

Let the two circles *ACB*, *ADB*, cut one another



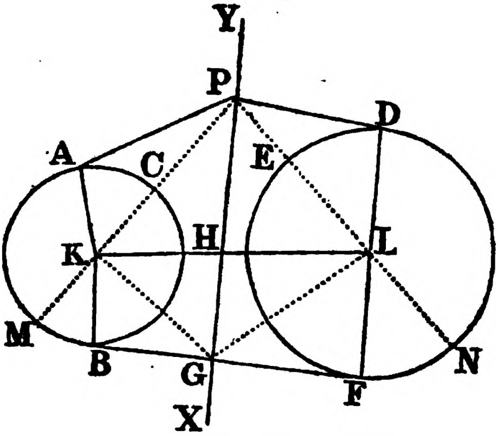
in *A* and *B*, and from any point *E*, without the circles, let there be drawn $\overline{EC}$ touching the circle *ACB*, and $\overline{ED}$ touching the circle *ADB*: If $EC = ED$, the points *E*, *A*, and *B*, are in the same straight line.

For join *E*, *A*; then shall $\overline{EA}$ produced pass through *B*; if not, let it pass otherwise, as $\overline{EAFG}$: Then (E. xxxvi. 3.) $\overline{FE} \times EA = EC^2$; also $\overline{GE} \times EA = ED^2$; and (*hyp.*) $EC^2 = ED^2$; $\therefore FE \times EA = GE \times EA$; $\therefore FE = GE$; *i. e.* the less of two straight lines is equal to the greater, which is impossible; $\therefore EA$, produced, cannot pass otherwise than through *B*; so that the three points *E*, *A*, and *B* are in the same straight line.

PROP. LXXIX.

101. PROBLEM. *Two circles being given, neither of which lies within the other, to draw a straight line, such that the tangents, to the two circles, drawn from any point of the line, shall be equal to one another.*

Let *ABC, DEF,* be two circles, neither of which lies within the other: It is required to draw a straight line, such that the tangents to the two circles, drawn from any point of the line, shall be equal to one another.



Find (E. i. 3.) the centres *K* and *L*, of the two given circles, and draw $\overline{KL}$; draw (Supp. lii. 3.) $\overline{BF}$ touching the two circles, on the same side, in *B* and *F*; bisect (E. xviii. 1.) *BF* in *G*; and through *G* draw (E. xii. 1.) $\overline{XY}$ perpendicular to *KL*: The tangents drawn to the two circles *ABC, DEF,* from any point in *XY* are equal to one another.

For take any point *P* in *XY*, and draw (E. xvii. 3.) from *P* the straight lines $\overline{PA}$ and $\overline{PD}$, touching the circles in *A* and *D* respectively; and draw $\overline{PK}$, $\overline{AK}$, $\overline{BK}$, $\overline{KG}$, $\overline{LG}$, $\overline{LD}$ and $\overline{LP}$.

And because (*constr.*) the angles at *H* are right angles; therefore (E. xlvii. 1.) $PK^2 + \overline{LG}^2 = \overline{PH}$, $+\overline{HK}^2 + \overline{LH}^2 + \overline{HG}^2$; and $\overline{PL}^2 + \overline{KG}^2 =$ the same four squares; ∴ $PK^2 + LG^2 = PL^2 + KG^2$; but (*constr.* and E. xviii. 3.) the ∠s *PAK, KBG, GFL,* and *LDP*, are right angles; ∴ (E. xlvii. 1.) $PK^2 + LG^2 = PA^2 + AK^2 + \overline{LF}^2 + \overline{FG}^2$; and $PL^2 + KG^2 = PD^2 + LF^2 + AK^2 + \overline{GF}^2$, because (*constr.*) $GF = \overline{GB}$; and $\overline{DL} = LF$, and $\overline{KB} = \overline{KA}$; ∴ $PA^2 + AK^2 + LF^2 + FG^2 = PD^2 + LF^2 + AK^2 + FG^2$; take away, therefore, from both, the squares of *AK*, of *LF* and of *FG*, and there remains $PA^2 = PD^2$; ∴ $PA = PD$.

102. Cor. 1. The difference of the squares of the distances of any point *P* in the line *XY* so drawn, from the centres *K* and *L*, of the two given circles, is equal to the difference of the squares of the two semi-diameters of the circles.

103. Cor. 2. If from any point *P* in the straight line *XY*, so drawn, any two straight lines be drawn, *PCM, PEN*, the one of them cutting the one of the given circles, the other the other, the rectangles, $\overline{MP} \times \overline{PC}$, $\overline{NP} \times \overline{PE}$, contained by the whole lines and the parts of them without the circles, shall be equal to one another.

For draw (E. xvii. 3.) from *P*, *PA* touching the circle *ABC*, and *PD* touching the circle *ADE*: Then (E. xxxvi. 3.) $MP \times PC = PA^2$; and $NP \times PE$, PD^2; but (Supp. lxxix. 3.) $PA^2 = PD^2$; ∴ $MP \times PC = NP \times PE$.

PROP. LXXX.

104. **PROBLEM.** *To find a point from which if straight lines be drawn to touch three given circles, none of which lies within another, the tangents so drawn shall be equal to one another.*

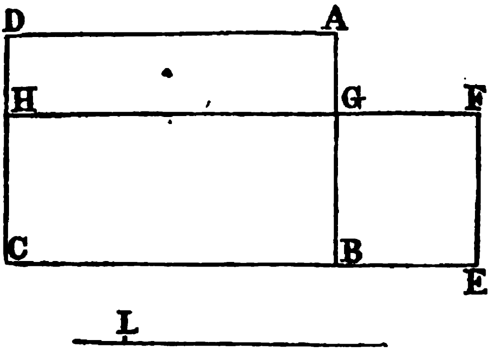
Draw (Supp. lxxix. 3.) the straight line which is the *locus* of equal tangents drawn to two of these given circles; draw likewise, the straight line, which is the *locus* of equal tangents drawn to the remaining circle and to either of the two circles first taken: It is manifest that the intersection of the two straight lines, so drawn, will be the point which was to be found.

105. COR. If from the point, thus found, any number of straight lines be drawn cutting the three given circles, the rectangles contained by the whole lines, so drawn, and the parts of them without the circles, shall (E. xxxvi. 3. and Supp. lxxx. 3.) be equal to one another.

PROP. LXXXI.

106. **PROBLEM.** *To divide a given straight line into two parts, so that the square of the one shall be equal to the rectangle contained by the other and a given straight line.*

Let *AB* and *L* be two finite straight lines: It is required to divide *AB* into two parts, so that the square of the one shall be equal to the rectangle contained by the other and by the given line L.



From B draw (E. xi. 1.) $\overline{BC}$ perpendicular to AB; make $BC=L$, and (E. xxxi. 1.) complete the $\square ABCD$; produce (Supp. lxxiii. 3. *Cor.*) CB to E, so that $\overline{CE} \times \overline{EB}$ may be equal to $\overline{AB} \times \overline{L}$; lastly upon $\overline{BE}$ describe (E. xlvi. 1.) the square $EFGB$: Then, AB is divided in G, so $\overline{BG}^2 = \overline{AG} \times L$.

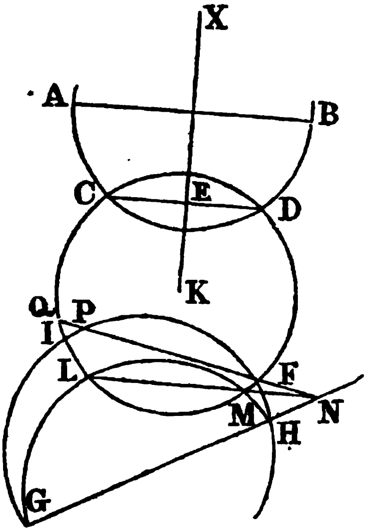
For produce $\overline{FG}$ to H; then (*constr.*) the rectangle $CE \times EB$, $= AB \times L$; but CF is the rectangle $CE \times EB$, because $EF = EB$; and CA is the rectangle $AB \times L$, because CB was made equal to L; therefore the rectangle $CF = CA$; take away the common part CG, and there remains $BF = HA$; and BF is the square of BG, and $HA = AG \times L$, because (*constr.* and E. xxiv. 1.) $\overline{AD} = BC$, which was made equal to L.

PROP. LXXXII.

107. THEOREM. *If a given circle be cut by any number of circles, which all pass through the same*

two given points without the given circle, the straight lines, joining the points of each of these intersections, are either all parallel, or all meet when produced in the same point.

Let *CDF* be the given circle; and, first, let the



circle *ACDB,* which passes through the two given points *A* and *B,* cut the circle *CDF* in *C* and *D*; let the straight line joining *C, D,* be parallel to $\overline{AB}$; then shall the straight line joining the points, in which any other circle that passes through *A, B,* cuts the circle *CDF,* be parallel to *AB* and $\overline{CD}$.

For, find (E. i. 3.) the centre *K* of the circle *CDF,* and from *K* draw (E. xii. 1.) $\overline{KEX}$ perpendicular to *CD*; therefore (E. iii. 3.) $\overline{KX}$ bisects *CD* at right angles; therefore (E. i. 3. *Cor.*) the centre of the circle *ACDB* is in *KX,* which (*hyp.* and E. xxix. 1.) cuts *AB* at right angles, and therefore bisects it; the centres, therefore, of all the circles that pass through A and B are (Supp. iii. 1. *Cor.* 3.) in KX; therefore (Supp. i. 3.) KX cuts all the straight lines, which join the intersections of these circles, with the given circle CDF, at right angles; therefore (E. xxviii. 1.) the straight lines joining the several pairs of intersections are parallel to one another and to AB.

But, secondly, let the circle $GLMH$, which passes through the two given points G, H, cut the given circle CDF in L and M; and let the straight line joining L and M be not parallel to AB; produce, therefore, $\overline{LM}$ to meet $\overline{GH}$ produced in N; and let any other circle $GIFH$, passing through G and H, cut the circle CDF in I and F; then are the points I, F and N in the same straight line.

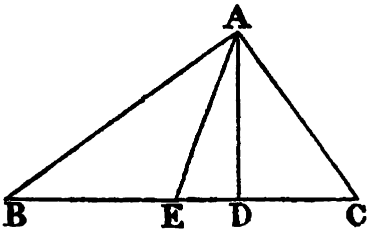
For join N, F, and if $\overline{NF}$, produced, do not pass through I, let it, if it be possible, pass otherwise, as $\overline{NFPQ}$: Then (E. xxxvi. 3. *Cor.*) $\overline{PN} \times NF = \overline{GN} \times \overline{NH}$; also $\overline{QN} \times NF = \overline{LN} \times \overline{NM}$, and $LN \times NM = GN \times NH$; $\therefore$ $QN \times NF = GN \times NH$; also $PN \times NF = GN \times NH$; $\therefore$ $QN \times NF = PN \times NF$; $\therefore$ $QN = PN$; that is the less is equal to the greater, which is impossible; $\therefore$ NF, when produced, cannot pass otherwise than through the point I, so that the three points I, F and N are in the same straight line.

PROP. LXXXIII.

108. THEOREM. *If a perpendicular be let fall from the right angle, of a right-angled triangle, on the hypotenuse, the rectangle contained by the hypotenuse, and either of the segments, into which it is*

divided by the perpendicular, is equal to the square of the side adjacent to that segment.

Let the ∠BAC, of the △ABC, be a right angle,



and from A let $\overline{AD}$ be drawn perpendicular to the hypotenuse BC: Then $\overline{CB}\times\overline{BD}=\overline{AB}^2$, and $\overline{BC}\times\overline{CD}=\overline{AC}^2$.

For if upon AC, as a diameter, a circle be described, it will pass (Supp. xxix. 1. *Cor.* 2.) through the point D, because (*hyp.*) the ∠ADC is a right angle; and (E. xvi. 3. *Cor.*) it will touch AB in A, because the ∠CAB is a right angle; therefore (E. xxxvi. 3.) $CB \times BD = AB^2$.

And, in the same manner, it may be shewn that $BC \times CD = AC^2$.

Prop. LXXXIV.

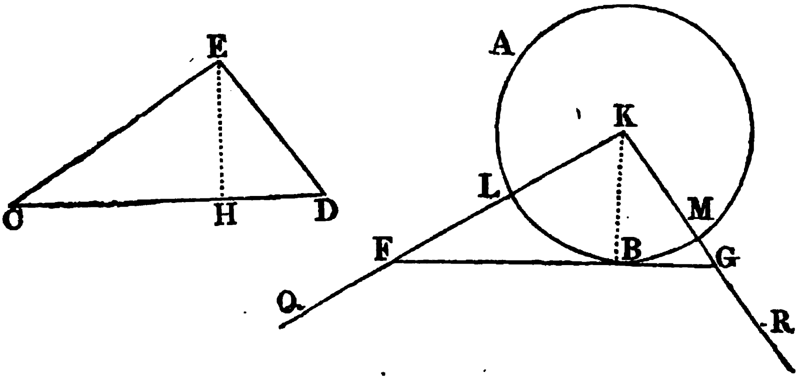
109. Theorem. *To draw a tangent to a circle, such, that the part of it intercepted between two straight lines, given in position, but of indefinite length, shall be equul to a given finite straight line:*

1st, *When the indefinite straight lines both pass through the centre of the circle.*

2dly, *When they are parallel to one another.*

3dly, *When they are not parallel, but are equidistant from the centre.*

Let *AB* be a given circle, and $\overline{CD}$ a given straight

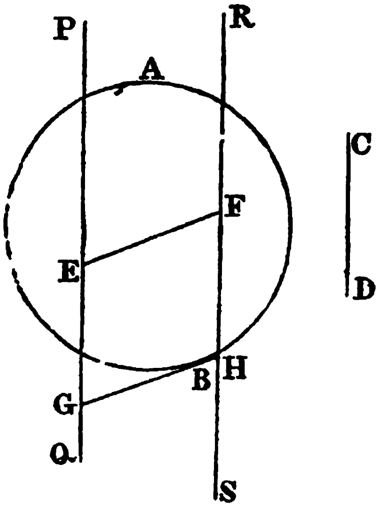


line; and first let *KQ* and *KR* be two given straight lines, of indefinite length, passing through the centre *K* of the circle: It is required to draw a straight line, touching the circle *AB*, so that the part of it intercepted between *KQ* and *KR*, shall be equal to *CD*.

Upon *CD* describe (Supp. lxi. 3.) a △*CED*, having its vertical ∠*CED* equal to the given ∠*QKR*, and its altitude *EH* = *KB*, the semi-diameter of the given circle; from $\overline{KQ}$ cut off $\overline{KF}=\overline{EC}$; and from *F* draw (E. xvii. 3.) the tangent $\overline{FBG}$ to the given circle: Then, the tangent $\overline{FG}$ = *CD*.

For let *FG* touch the circle in *B*, and join *K*, *B*; And since (E. xviii. 3.) the ∠*KBF* is a right angle, as is also (*constr.*) the ∠*EHC*, and that (*constr.*) *EC* = *KF*, and *EH*=*KB*, therefore (Supp. lxxiii. 1.) the ∠*ECH* = ∠*KFB*; but (*constr.*) the ∠*CED* = = ∠*FKG*; and the side *EC* of the △*ECD*, is equal to the side *KF*, of the △*KFG*; therefore (E. xxvi. 1.) *FG* = *CD*.

Secondly, let *AB* be the given circle, and *PQ*,

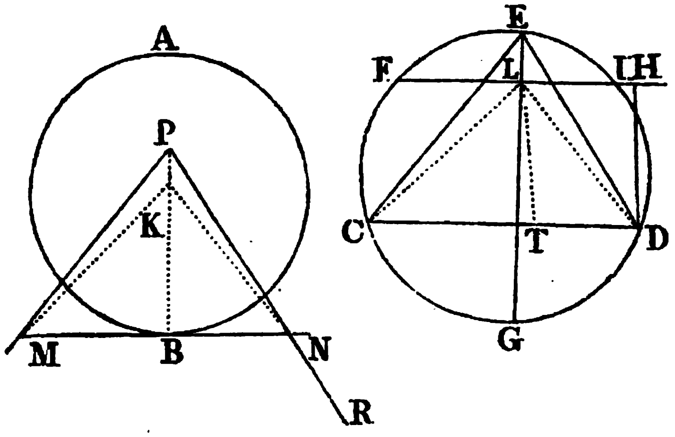


RS, two indefinite but parallel straight lines: It is required to draw a tangent to the circle *AB*, such that the part of it intercepted between $\overline{PQ}$ and $\overline{RS}$ shall be equal to the given straight line *CD*.

Take any point *E* in either of the two parallel straight lines, as *PQ*, and from the centre *E*, at a distance equal to *CD*, describe a circle cutting *RS* in *F*; join *E*, *F*; $\therefore EF = CD$; lastly, draw (Supp. viii. 3.) the straight line *GH*, touching the circle *AB*, and parallel to *EF*; since, therefore, *EGHF* is a parallelogram, *GH* (E. xxxiv. 1.) $= EF$; and *EF* was made equal to *CD*; therefore the tangent $GH = CD$.

Thirdly, let the two indefinite straight lines *PQ*, *PR*, which meet in *P*, be equidistant from the centre *K*, of the circle *AB*: It is required to draw a straight line touching the circle *AB*, so that the part of it intercepted between *PQ* and $\overline{PR}$ shall be equal to the given straight line *CD*.

Join *P*, *K*; upon *CD* describe (E. xxxiii. 3.) segment of a circle *CED*, capable of containing an angle equal to the given $\angle QPR$, and complete the circle *CEDG*; from *D* draw (E. xi. 1.) $\overline{DH}$ perpendicular to *CD*, and make $DH = KB$ the semidiameter of the given circle *AB*; through *H* draw (E. xxxi. 1.) $\overline{HIF}$ parallel to $\overline{DC}$; bisect (E. xxx. 3.) $\overset{\frown}{FGI}$ in *G*; from *G* draw (Supp. lxxv. 3.) $\overline{GLE}$, so that $\overline{LE} = \overline{KP}$; join *E*, *C* and *E*, *D*; from *PQ* cut off $\overline{PM} = \overline{EC}$; and from *M* draw (E. xvii. 3.) $\overline{MBN}$ touching the circle in *B*: The tangent $MN = CD$.



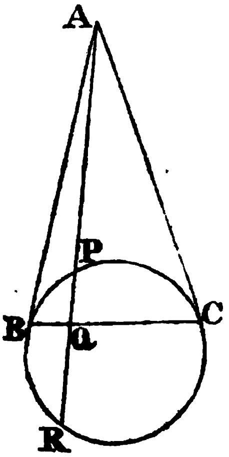
For join *C*, *L*, and *D*, *L*, and *K*, *M*, and *K*, *N*, and *K*, *B*; and draw (E. xii. 1.) $\overline{LT}$ perpendicular to *CD*; ∴ *LTDH* is a parallelogram, and (E. xxxiv. 1.) $LT = \overline{HD}$; and *HD* (*constr.*) $= \overline{KB}$; ∴ $LT = KB$: Again, because (*constr.* and Supp. xxxv. 3.) $\overset{\frown}{CG} = \overset{\frown}{DG}$, therefore (E. xxvii. 3.) the $\angle CEG = \angle DEG$; therefore the $\angle CEL$ is the half of the $\angle CED$; and because (*hyp.*) *PQ* and *PR* are equidistant from the centre *K* of the circle *AB*, therefore the $\angle QPK$, or *MPK*, is the half of the $\angle QPR$, which (*constr.*) is equal to the ∠ *CED*; therefore the ∠ *MPK* = ∠ *CEL*, and the two sides *MP*, *PK*, of the △ *PKM*, are equal (*constr.*) to the two sides *CE*, *EL* of the △ *ELC*, each to each; therefore (E. iv. 1.) *KM*=*LC*, and the ∠ *PMK*, = ∠ *ECL*; and because in the two right-angled △s *KBM*, *LTC*, *KM*=*LC*, and *KB*=*LT*; therefore (Supp. lxxiv. 1.) the ∠ *KMB* = ∠ *LCT*; and it has been shewn that the ∠ *PMK* = ∠ *ECL*; therefore the whole ∠ *PMN* is equal to the whole ∠ *ECD*; also (*constr.*) the ∠ *MPN* = ∠ *CED*, and the side *PM*, of the △ *PMN*, is equal to the side *EC*, of the △ *ECD*: therefore (E. xxvi. 1.) $\overline{MN}=CD$.

PROP. LXXXV.

110. THEOREM. *If from the intersection of any two tangents to a circle, any straight line be drawn, cutting the chord which joins the two points of contact and again meeting the circumference, it shall be divided by the circumference and the chord into three segments, such, that the rectangle contained by the whole line and the middle part, shall be equal to the rectangle contained by the extreme parts.*

From the intersection *A* of two straight lines *AB* and *AC* which touch the circle *BCR* in the points *B* and *C*, let there be drawn any straight line *APR*, cutting the circumference of the circle in *P* and *R*, and $\overline{BC}$ in *Q*: Then $\overline{AR} \times \overline{PQ} = \overline{AP} \times \overline{QR}$.

For since (Supp. xix. 3. *Cor.* 1.) $AB=AC$, the $\triangle ABC$ is isosceles, and



therefore (Supp. iii. 2.) $\overline{AQ}^2+\overline{BQ}\times\overline{QC}=\overline{AB}^2$;
and (E. xxxv. 3.) $BQ\times QC=PQ\times QR$;
also (E. xxxvi. 3.) $AB^2=AP\times AR$;
$\therefore\ AQ^2+PQ\times QR=AP\times AR$;
i. e. (E. i. 2.) $AQ\times AP+AQ\times PQ+PQ\times QR=AP\times AQ+AP\times QR$;

From these equals take away the common rectangle $AQ\times AP$, and there remains

$$AQ\times PQ+QR\times PQ=AP\times QR;$$

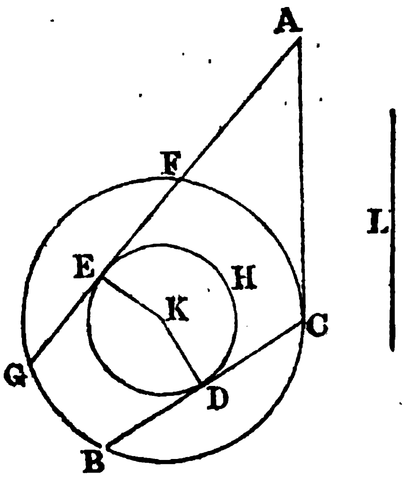
i. e. (E. i. 2.) $AR\times PQ=AP\times QR$.

PROP. LXXXVI.

111. PROBLEM. *To make a rectangle which shall be equal to a given square, and have the difference between its two adjacent sides equal to a given straight line.*

Let AC be the side of a square, and let L be a finite straight line: It is required to describe a rectangle which shall be equal to the square of *AC*, and shall have the difference between its two adjacent sides equal to *L*.

Describe a circle *CBGF* with a diameter not less



than *L*, and having its centre in a perpendicular to $\overline{AC}$ at the point *C*; the circle *CBGF* is, therefore, (E. xvi. 3.) touched by *AC* in *C*; from the centre *C*, at a distance = *L*, describe a circle cutting the circumference of *CBGF* in *B* and draw $\overline{CB}$; ∴ *CB* $= \overline{L}$; from the centre *K* of the circle *CBGF*, draw (E. xii. 1.) $\overline{KD}$ perpendicular to $\overline{BC}$; and from *K*, as a centre, at the distance *KD*, describe the circle *DEH*, which therefore (E. xvi. 3.) touches *BC* in *D*; lastly, from *A* draw (E. xvii. 3.) a straight line *AFG* touching the circle *DEH* in *E*, and let *AG* cut the circumference of *CBFG* in *F* and *G*: Then is the rectangle contained by $\overline{GA}$ and $\overline{AF}$ that which was to be described.

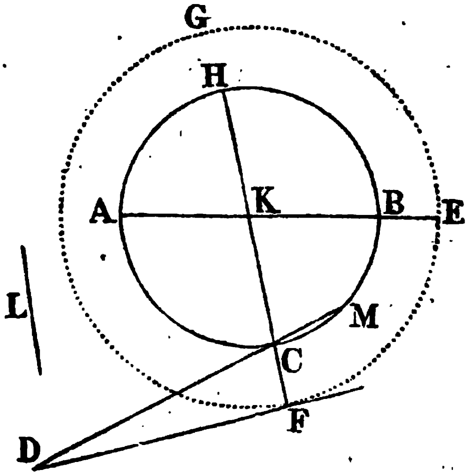
For join K, E; therefore (*constr.* and E. xviii. 3.) the angles at E are right angles; $\therefore$ $\overline{GF}$, which is the difference of GA and AF, is (E. xiv. 3.) equal to BC, which was made equal to L; $\therefore$ $GF=L$; also, since AC touches the circle $CBGF$, therefore (E. xxxvi. 3.) $GA\times AF=AC^2$.

112. Cor. Hence, and from E. xiv. 2. a rectangle may be found which shall be equal to a given rectangle, and which shall have the difference between its two adjacent sides equal to a given straight line.

Prop. LXXXVII.

113. Problem. *From a given point without a circle, to draw a straight line cutting the circle, so that the rectangle contained by the part of it without, and the part within, the circle, shall be equal to a given square.*

Let ABC be a circle, D a point without the circle,



and L a finite straight line: It is required to draw, from D, a straight line cutting the circle

ABC, so that the rectangle contained by the part of it without, and the part of it within, the circle, shall be equal to the square of *L*.

Find (E. i. 3.) the centre *K* of the circle *ABC*; take any diameter *AKB*, and produce it (Supp. lxvii. 3.) to *E*, so that $\overline{AB} \times \overline{BE}$ = the square of *L*; from the centre *K*, at the distance *KE*, describe the circle *EFG*; from *D* draw (E. xvii. 3.) the straight line *DF* touching the circle *EFG* in *F*; join *K*, *F*, and let $\overline{KF}$ cut the circumference of *ABC* in *C*; lastly, draw $\overline{DC}$, and produce it to meet the circumference of *ABC* again in *M*: Then shall $DC \times \overline{CM}$ be equal to the square of *L*.

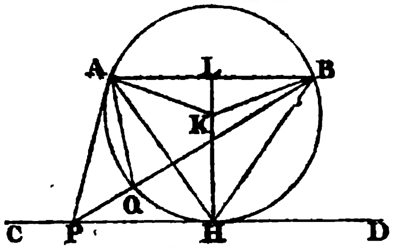
For, produce $\overline{CK}$ to meet the circumference of *ABC* again in *H*; then (E. xv. Def. 1.) $\overline{HC} = \overline{AB}$, and $\overline{CF} = BE$; ∴ $HC \times CF = AB \times BE$; but (Supp. lxix. 3.) $DC \times CM = HC \times CF$, and (*constr.*) $AB \times BE$ = the square of *L*; ∴ $DC \times CM$ = the square of *L*.

PROP. LXXXVIII.

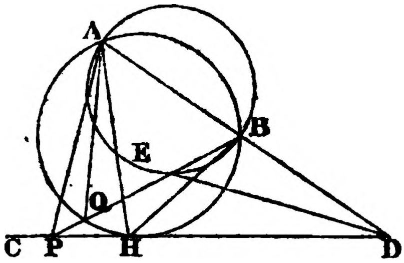
114. **PROBLEM.** *To describe a circle which shall touch a given straight line, and pass through two given points, both on the same side of the given line, and in the same plane with it.*

Let *CD* be a straight line, and *A*, *B*, two points without it, both on the same side of *CD*; it is required to draw a circle through *A* and *B*, which shall touch *CD*.

Join *A*, *B*; and first, let *AB* be parallel to *CD*: Bisect (E. x. 1.) *AB* in *L*; through *L* draw (E. xi. 1.) *LH* perpendicular to *AB* or *CD*; join *A*, *H*; at the point *A*, in *HA*, make (E. xxiii. 1.) the angle *HAK* equal to the angle *AHK*, and join *K*, *B*; then (E. vi. 1.) *KH* is equal to *KA*, and (E. iv. 1.) *KA* is equal to *KB*; from the centre *K*, at the distance *KA*, or *KB*, or *KH*, describe the circle *AHB*; it shall pass through the three points *A*, *H*, and *B*, and (E. xvi. 3. *Cor.*) shall touch *CD* in *H*.



But if *AB* be not parallel to *CD*, let *AB*, produced, meet *CD* in the point *D*. Upon *AB* as a diameter describe the circle *ABE*, and from *D* draw (E. xvii. 3.) the straight line *DE* touching it in *E*; from *DC* cut off *DH* (E. iii. 1.) equal to *DE*, and describe (E. v. 4.) the circle *AHB* passing through the three points *A*, *H*, and *B*. The circle *AHB*, which passes through *A* and *B*, touches *CD* in *H*.



For (E. xxxvi. 3.) the rectangle contained by *AD* and *DB* is equal to the square of *DE*, and, therefore, is equal also to the square of *DH*, because *DH* was made equal to *DE*; wherefore (E. xxxvii. 3.) the circle *AHB* touches *CD* in *H*.

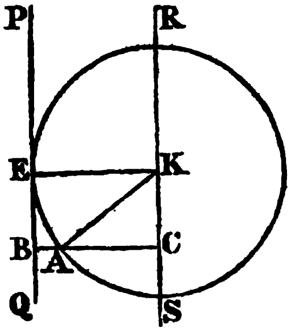
115. Cor. *AB* subtends a greater angle at the point *H*, in the straight line *CD*, than at any other point whatever in *CD*.

For, let *P* be any other point in *CD*; *P* is without the circle *AHB*; join *A*, *P*, and *B*, *P*; let $\overline{BP}$ cut the circle in *Q*; also join *A*, *Q*. The angle *AHB* is equal (E. xxi. 3.) to the angle *AQB*; but the exterior angle *AQB* is greater (E. xvi. 1.) than the interior opposite angle *APB*; wherefore, also, *AHB* is greater than *APB*.

Prop. LXXXIX.

116. Problem. *To describe a circle which shall have its centre in a given straight line, which shall pass through a given point, and shall, also, touch another given straight line.*

Let *A* be a point, between two straight lines given in position; and first let the two straight lines *PQ*, *RS*, between which *A* is posited, be parallel to one another: It is required to describe a circle, which shall have its centre in $\overline{RS}$, which shall pass through the given point *A*, and touch $\overline{PQ}$.



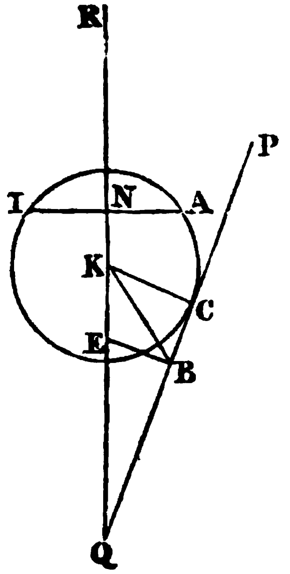
Through *A* draw (E. xii. 1.) $\overline{BAC}$ perpendicular to *PQ*; from *A*, as a centre, at a distance equal to *CB*, describe a circle, and let it cut *RS* in *K*; from *K* draw *KE* (E. xxxi. 1.) parallel to *CB*; therefore the figure *EC* is a parallelogram, and, therefore, (E. xxxiv. 1. and E. xxix. 1.) $\overline{KE} = \overline{CB}$, and the $\angle KEB$ is a right angle; also, since (*constr.*) $\overline{KA} = CB$, and that *CB*, as hath been shewn, is equal to *KE*; $\therefore KE = KA$; and therefore a circle described from *K* as a centre, at the distance *KE*, will pass through *E* and *A*; and, because the $\angle KEB$ is a right angle, it will (E. xvi. 3.) touch *PQ* in *E*.

Secondly, let *K* be a given point in the given straight line *RQ* which meets another given straight line *PQ* in *Q*; and let it be required to describe a circle which, having its centre in $\overline{RQ}$, shall pass through *K*, and which shall touch *PQ*.

From *K* draw (E. xii. 1.) $\overline{KC}$ perpendicular to *PQ*; bisect (E. ix. 1.) the $\angle CKQ$ by $\overline{KB}$, and from *B* draw (E. xi. 1.) $\overline{BE}$ perpendicular to *PQ*; therefore (E. xxviii. 1.) *BE* is parallel to $\overline{CK}$; therefore (E. xxix. 1.) the $\angle CKB = \angle KBE$; but (*constr.*) the $\angle EKB = \angle CKB$; therefore the $\angle EKB = \angle KBE$; therefore (E. vi. 1.) $EK = EB$; and therefore a circle described from the centre *E*, at the distance *EK*, will pass through *B*, and (E. xvi. 3.) touch *PQ* in *B*, because (*constr.*) the $\angle EBQ$ is a right angle.

Lastly, let the given point *A* be between two given straight lines *PQ* and *RQ* which meet in *Q*; and let it be required to describe a circle which shall have its centre in *RQ*, which shall pass through *A*, and touch *PQ*.

From *A* draw (E. xii. 1.) $\overline{AN}$ perpendicular to *RQ*, and produce *AN* to *I*, so that $\overline{NI} = \overline{NA}$; describe



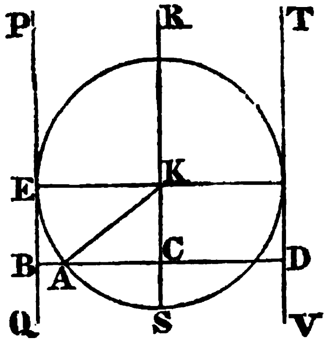
(Supp. lxxxviii. 3.) a circle which shall pass through *A* and *I* and touch *PQ*; and since *RQ* bisects $\overline{AI}$ at right angles, the centre of the circle will (E. i. 3. *Cor.*) be in *RQ*.

PROP. XC.

117. PROBLEM. *To describe a circle which shall touch two given straight lines, and pass through a given point between them.*

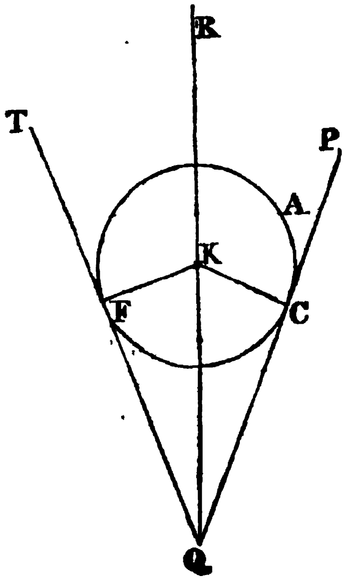
Let *A* be a point, between two straight lines *PQ*, *TV*, and, first, let *PQ* be parallel to *TV*: It is required to describe a circle, which shall pass through *A* and touch both $\overline{PQ}$ and $\overline{TV}$.

Through *A* draw (E. xii. 1.) $\overline{BAD}$ perpendicular to *PQ*, and therefore (E. xxix. 1.) also perpendicular to *TV*; bisect (E. x. 1.) *BD* in *C*, and through *C* draw (E. xxxi. 1.) $\overline{RCS}$ parallel to *PQ*, and therefore



(E. xxx. 1.) also parallel to *TV*; lastly, describe (Supp. lxxxix. 3.) a circle which shall have its centre in *RS*, shall pass through *A*, and touch *PQ*: It will also, since its semi-diameter is equal to *CB* or *CD*, touch *TV*.

Secondly, let the given point *A* be between two straight lines *TQ*, and *PQ*, which meet in *Q*:



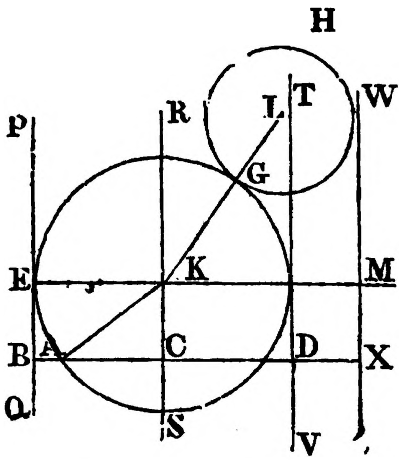
bisect (E. ix. 1.) the ∠ *TQP* by $\overline{RQ}$, and since

(E. xxvi. 1.) the perpendicular distances of any point in *RQ*, from $\overline{TQ}$, and *PQ*, are equal to one another, it is manifest, that, if (Supp. lxxxix. 3.) a circle be described having its centre in *RQ*, passing through *A*, and touching either of the two lines *TQ*, *PQ*, it will touch the other also.

Prop. XCI.

118. **Problem.** *To describe a circle which shall touch two given straight lines, and also touch a given circle, which does not lie wholly without the two given straight lines.*

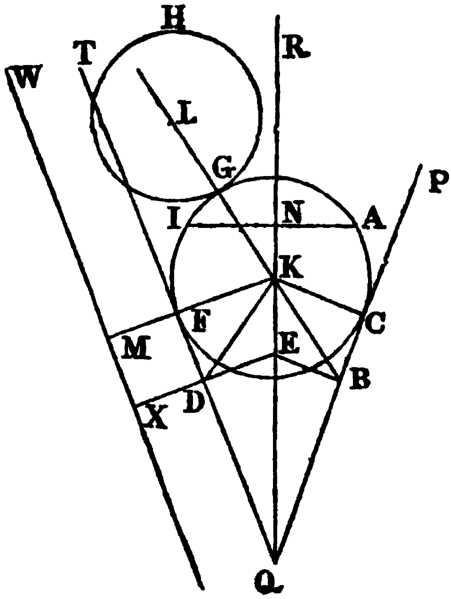
Let $\overline{PB}$ and $\overline{TD}$ be two straight lines, and



let *HG* be a circle, which does not lie wholly without *PB* and *TD*: It is required to describe a circle which shall touch both *PB* and *TD*, and which shall also touch the circle *GH*.

Find (E. i. 3.) the centre *L* of the circle *GH*; from *D* draw (E. xi. 1.) $\overline{DX}$ perpendicular to *TD*, and make it equal to the semi-diameter of *HG*; through *X* draw (E. xxxi. 1.) $\overline{XW}$ parallel to *TD*;

also, as in Supp. xc. 3. draw $\overline{RK}$, equidistant from

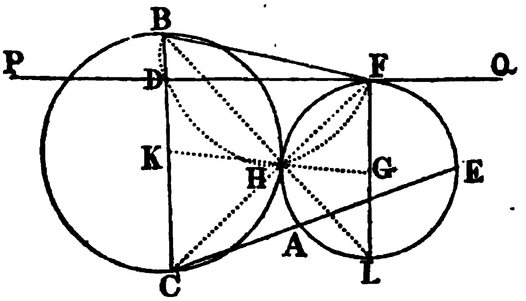


PB and TD; describe (Supp. lxxxix. 3.) a circle which shall have its centre in RK, which shall pass through L, and touch $\overline{WX}$; let K be the centre of the circle, so described, and let it touch WX in X; join K, M and K, L; and let $\overline{KM}$ and $\overline{KL}$ cut TD, and the circumference of GH, in F and G, respectively: Then, since (E. xviii. 3.) the $\angle\, KMW$ is a right angle, and that (*constr.*) WX is parallel to TD; therefore the $\angle\, KFT$ is also a right angle; and, because (*constr.*) MD is a parallelogram, (E. xxxiv. 1.) $\overline{MF} = \overline{XD}$; but (*constr.*) $XD = \overline{LG}$; $\therefore$ $MF = LG$; and (*constr.* and E. xv. Def. 1.) $KM = KL$; $\therefore$ $\overline{KF} = \overline{KG}$, and a circle described from the centre K, at the distance KF, will (E. xvi. 3. *Cor.*) touch TD in F, will pass through G, and (Supp. vi. 3.) will touch the circle HG in G.

PROP. XCII.

119. **PROBLEM.** ***To describe a circle which shall touch both a given circle, and a given straight line, and which shall, also, pass, first, through a given point without the given circle; and, secondly, through a given point within the circle.***

Let *BCH* be a circle, *PQ* a straight line, and first let the given point *A* be without the circle: It is required to describe a circle which shall pass through *A*, and which shall touch both $\overline{PQ}$ and the circle *BCH*.



Find (E. i. 3.) the centre *K* of the circle *BCH*, and draw (E. xii. 1.) the diameter *BDKC* perpendicular to *PQ*; join *C*, *A**, and produce *CA* to *E* (Supp. lxvii. 3. *Cor.*) or divide it (Supp. lxxi. 3. *Cor.* 3.) so that $\overline{EC} \times \overline{CA} = \overline{BC} \times \overline{CD}$; describe

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* If $\overline{AC}^2 > \overline{BC} \times \overline{CD}$, then $\overline{CA}$ must be divided into two parts, so that the rectangle contained by *AC* and the segment toward *C* shall be equal to $BC \times CD$. Also, in this application of Supp. lxvii. 3, *CA* must first be produced, so that the rectangle contained by *CA* and the *part* produced, shall be of the given magnitude; and then from the whole line, *CE* must be cut off equal to the part produced.

(Supp. lxxxviii. 3.) a circle *AEF*, which shall pass through *A* and *E*, and touch $\overline{PQ}$: It shall also touch the circle *BCH*.

For, let the circle *AEF* touch *PQ* in *F*; find its centre *G*; and draw the diameter *FGL*, which (*constr.* E. xviii. 3. and E. xxviii. 1.) is parallel to *BC*; join, *B*, *F*, and *C*, *F*; and let $\overline{CF}$ cut the circumference of *BCH* in *H*; join, also, *B*, *H* and *F*, *H* and *K*, *H*; and let $\overline{KH}$ meet *FL* in *G*; upon *BF* as a diameter, describe the circle *BDHF*, which, because the ∠s *BDF*, *BHF* (*constr.* and E. xxxi. 3.) are right angles, will pass (Supp. xxix 1. *Cor.* 2.) through *D* and *H*; therefore (E. xxxvi. 3. *Cor.*) $BC \times CD = \overline{FC} \times \overline{CH}$; but (*constr.*) $BC \times CD = EC \times CA$; $\therefore FC \times CH = EC \times CA$, and, therefore, the point *H* is in the circumference of the circle *AEF*; otherwise (E. xxxvi. 3. *Cor.*) the greater of two rectangles would be equal to the less. The point is, therefore, common to both the circles *AEF*, *BCH*.

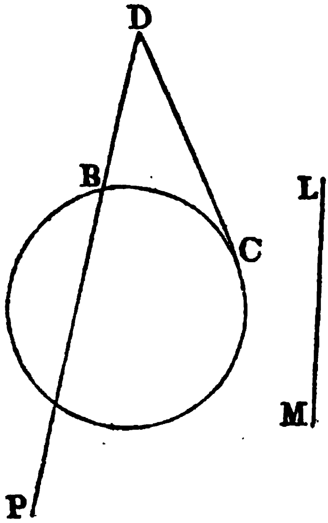
And, since (*constr.*) $\overline{FL}$ is parallel to *BC*, therefore (E. xxix. 1.) the $\angle GFH = \angle HCB$; but, since (*constr.* and E. xxxi. 3.) the $\angle BHC$ is a right angle, the $\angle HCB + \angle CBH =$ (E. xxxii. 1.) a right angle; therefore the $\angle GFH + \angle KBH =$ a right angle; that is, (E. xv. Def. 1. and E. v. 1.) the $\angle GFH + \angle KHB$ = a right angle; and (*constr.* and E. xxxi. 3.) the $\angle BHF$ is a right angle; therefore (E. xiii. 1.) the $\angle KHB + \angle GHF =$ a right angle; therefore, the $\angle GFH = \angle GHF$, and (E. vi. 1.) $GF = GH$: But *G* is in the diameter of the circle *AEF*; therefore (E. vii. 3.) *G* is the centre of the circle *AEF*, which therefore (Supp. vi. 3.) touches the circle *BCH* in *H*.

And, in a similar manner, the problem may be solved, when it admits of a solution, if the given point be within the given circle: It is manifest, however, that, in this latter case, the given straight line which is to be touched cannot lie wholly without the given circle.

PROP. XCIII.

120. PROBLEM. *In a straight line of indefinite length, but given in position, which cuts a given circle, to find a point, from which if a straight line be drawn to touch the circle, it shall be equal to a given finite straight line.*

Let *LM* be a finite straight line, *PAB* a straight line given in position, but indefinite in length, cutting the given circle *ABC* in *A* and *B*: It is required to find a point in *PB*, from which, if a tangent be drawn to the circle *ABC*, it shall be equal to *L*.

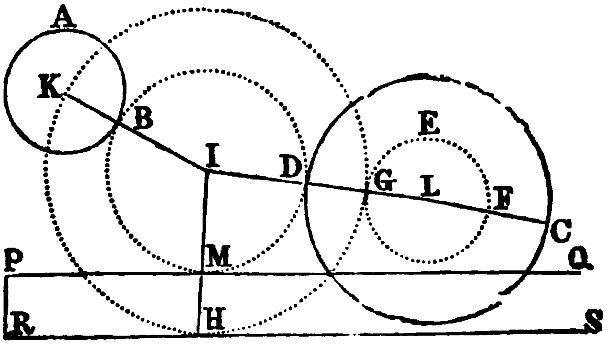


Produce (Supp. lxxiii. 3.) *AB* to *D* so that $\overline{AD} \times \overline{DB} = \overline{LM}^2$; and from the centre *D*, at a distance = *LM*, describe a circle cutting the circumference of the circle of *ABC* in *C*; draw $\overline{DC}$; ∴ $DC = LM$; therefore but (*constr.*) $AD \times DB = LM^2$; ∴ $AD \times DB = DC^2$; ∴ (E. xxxvii. 3.) *DC* touches the circle *ABC*, in *C*; and (*constr.*) it is equal to *LM*, and is drawn from a point *D* in the given indefinite straight line *PAB*.

PROP. XCIV.

121. PROBLEM. *To describe a circle that shall touch a given straight line, and that shall also touch two given circles.*

Let *AB* and *CD* be the two given circles, and *PQ* the given straight line; and first, let neither of the two given circles lie within the other: It is required to describe a circle which shall touch both the given circles *AB* and *CD*, and which shall also touch $\overline{PQ}$.



Find (E. i. 3.) the centres *K* and *L* of the circles *AB* and *CD*; and if the circles be unequal, let *CD* be the greater; from any semi-diameter, as *LC*, of the greater, cut off *CF* equal to a semi-diameter of the less circle; from the centre *L*, at the distance *LF* describe the circle *FGE*; from any point *P*, in *PQ*, draw (E. xi. 1.) $\overline{PR}$ perpendicular to *PQ*, and make *PR* also equal to the semi-diameter of the less circle *AB*; through *R* draw (E. xxxi. 1.) *RS* parallel to *PQ*; describe (Supp. xcii. 3.) a circle *KHG*, passing through the point *K*, touching $\overline{RS}$, in *H*, and touching the circle *EGF*, in *G*; let *I* be the centre of the circle *KHG*; join *K*, *I*, and *L*, *I*, and *I*, *H*; therefore (E. xviii. 3.) the ∠ *IHR* is a right angle, and ∴ (*constr.* and E. xxix. 1.) the exterior ∠ *IMP* is, also, a right angle; and the figure *PH* is a parallelogram; therefore (E. xxxiv. 1.) $\overline{MH} = PR$; and (*constr.*) $PR = \overline{KB}$ or $\overline{DG}$; ∴ $MH = KB$ or *DG*; ∴ $\overline{IB}$, $\overline{IM}$ and $\overline{ID}$ are all equal; and (E. xvi. 3. *Cor.* and Supp. vi. 3.) a circle described from the centre *I*, at the distance *IM*, will touch *PQ* in *M*, the circle *AB* in *B*, and the circle *CD* in *C*.

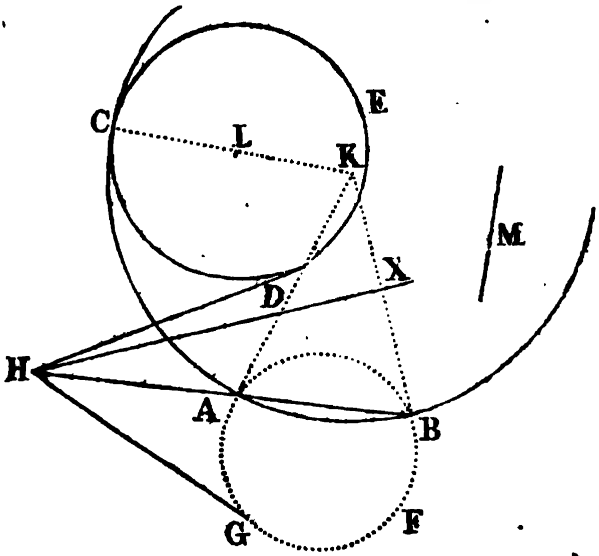
But if the two circles *AB*, *CD*, be equal to one another, find, as before, their centres *K*, and *L*, and draw *RS* at a perpendicular distance from *PQ* equal to the semi-diameter of *AB* or *CD*: Then, if (Supp. lxxxviii. 3.) a circle be described passing through *K* and *L*, and touching *RS*, it is evident, that its centre will be the centre of the circle which is to be described, and its semi-diameter will be found, as in the former case, by joining that centre and the centre of either of the two equal and given circles.

And, in a similar manner, the problem may be solved, when it admits of a solution, if the two given circles do not lie without one another.

Prop. XCV.

122. **Problem.** ***To describe a circle which shall touch a given circle, and pass through two given points, either both without the circle, or both within it.***

Let *A*, *B*, be two points, and *CDE* a circle: It is required to describe a circle which shall pass through *A* and *B*, and which shall also touch the circle *CDE*.



First, let the two given points, *A* and *B*, be without the circle *CDE*: And if *A* and *B* be equally distant from the centre of *CDE*, it is manifest (Supp. vi. 3.) that a circle described (Supp. v. 1. *Cor.*) so as to pass through the two given points, and through the extremity of a diameter of the given circle drawn perpendicular to the straight line joining those points, will touch the given circle.

But if the points, *A* and *B*, be not equally distant from the centre of the circle *CDE*, take any point *F*, without the circumference of *CDE*, and through *A*, *B* and *F*, describe (Supp. v. 2. *Cor.*) the circle *AFB*; draw (Supp. lxxix. 3.) $\overline{HX}$, so that the straight lines which are drawn from any point of it, touching the two circles *CDE*, *AGB*, shall be equal to one another, and let $\overline{BA}$, produced, meet *HX* in *H*; from *H* draw (E. xvii. 3.) $\overline{HC}$ and $\overline{HD}$, touching the circle *CDE* in *C* and *D*; lastly, describe (Supp. v. 1. *Cor.*) two circles, the one passing through *B*, *A*, and *C*, and the other through *B*, *A*, *D*; the circles so described shall touch the given circle *CDE*, in the points *C* and *D*, respectively.

For, from *H* draw (E. xvii. 3.) $\overline{HG}$, touching the circle *AGB* in *G*: Then (E. xxxvi. 3.) $\overline{BH} \times \overline{HA} = \overline{HG}^2$; but (*constr.*) $HG = HC$; $\therefore BH \times HA = HC^2$; $\therefore$ (E. xxxvii. 3.) *HC* touches the circle described through *B*, *A* and *C*; and (*constr.*) it also touches the circle *CDE*; $\therefore$ (E. iii. Def. 3.) the circle *BAC* which passes through *A* and *B*, touches the circle *CDE* in *C*.

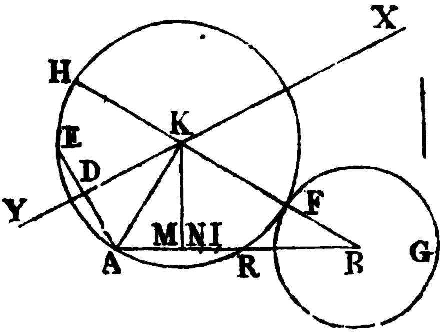
In the same manner it may be shewn, that the circle described so as to pass through *A*, *B* and *D*, touches the circle *CDE* in *D*: And by a like construction may the problem be solved, when the two given points are both within the given circle.

Prop. XCVI.

123. Problem. *To find a point in a straight line, given in position, from which if two straight lines be drawn to two given points, without the given*

line, they shall have, first, their difference, and, secondly, their aggregate, equal to a given finite straight line.

Let A, B be two points, XY a straight line of indefinite length, but given in position; and let C be a finite straight line: It is required to find a point in $\overline{XY}$, from which if two straight lines be drawn to A and B, they shall have, first, their difference equal to C.



From A draw (E. xi. 1.) $\overline{AD}$ perpendicular to XY; produce AD to E, and make $\overline{DE} = AD$; from the centre B, at a distance equal to C, describe the circle FG; also, describe (Supp. xcv. 3.) a circle EAF, which shall pass through E and A, and which shall touch the circle GF, in F; let K be the centre of the circle EAF, which centre (E. i. 3. *Cor.*) is in XY: Then is K the point which was to be found.

For join K, A and K, B; therefore (E. xi. 3. or E. xii. 3.) $\overline{KB}$ passes through the point of contact F; and (E. xv. Def. 1.) $\overline{KA} = \overline{KF}$; $\therefore KB - KA = \overline{BF}$; and (*constr.*) $BF = \overline{C}$; $\therefore KB - KA = C$.

And, by a like construction, may a point be found in XY, from which if two straight lines be drawn, to A and B, their aggregate shall be equal to a given straight line.

But, in this case, the two points A and E must fall within the circle described from the centre B, at a distance equal to that given line; otherwise, the problem is impossible.

124. Cor. 1. Let AB be (E. x. 1.) bisected in I, let (E. xii. 1.) KM be drawn perpendicular to AB, and let the circumference EAF cut AB in A and R, and BK produced in H: Then it is manifest, (*constr.* and E. iii. 3.) that $2IM=BR$; and (E. xxxvi. 3. *Cor.*) $\overline{AB}\times\overline{BR}=\overline{HB}\times BF$; *i. e.* $2AB\times\overline{IM}=HB\times BF$, or $\overline{HF}\times BF+BF^2$ (E. iii. 2.)

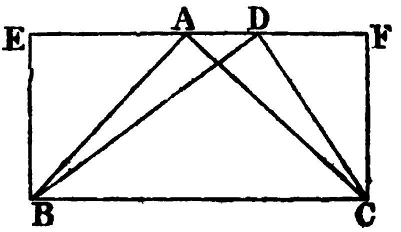
Let now IN be taken in IM (Supp. lxvii. 3.) so that $2AB\times\overline{IN}=BF^2$; therefore, if $2AB\times IN$ be taken from $2AB\times IM$, and if BF^2 be taken from $HF\times BF+BF^2$, there will remain $2AB\times\overline{NM}=HF\times BF$ or $2AK\times BF$; $\therefore$ $AB\times NM=\overline{AK}\times BF$.

125. Cor. 2. There is only one point K, in YX, from which if straight lines be drawn to A and B, their difference shall be equal to the given line C.

Prop. XCVII.

126. Problem. *The base and the altitude of a triangle being given, together with the aggregate, or the difference, of the two remaining sides, to construct the triangle.*

Let $\overline{BC}$ be the base of a triangle, and BE, drawn perpendicular to BC, equal to its altitude: It is required to construct a triangle, which shall have BC for its base, its altitude equal to $\overline{BE}$ and,



first, the aggregate of its two remaining sides of a given length.

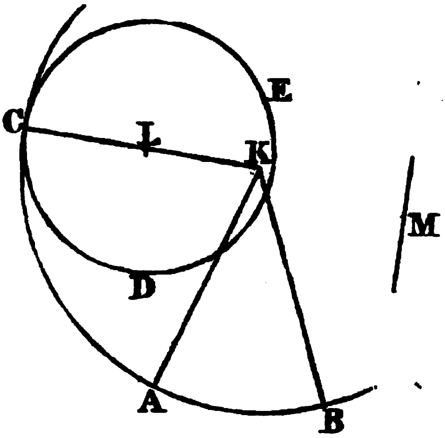
Through E draw (E. xxxi. 1.) $\overline{EF}$ parallel to BC; find (Supp. xcvi. 3.) a point A from which, if $\overline{AB}$ and $\overline{AC}$ be drawn, their aggregate shall be equal to the given aggregate; if, therefore, A, B and A, C be joined, it is manifest that ABC is the triangle, which was to be described.

And, in the same manner, by the help of Supp. xcvi. 3., may the problem be solved if the difference, instead of the aggregate of the two sides of the triangle, be given.

Prop. XCVIII.

127. Problem. *Three points being given, to find a fourth, from which if straight lines be drawn to the other three, two of them shall be equal, and the difference between either of these and the third shall be equal to a given straight line.*

Let A, B and L be three points, and M a finite straight line: It is required to find a fourth point, from which, if three straight lines be drawn to L, A, and B, two of them shall be equal, and the difference between either of these and the third shall be equal to *M*.



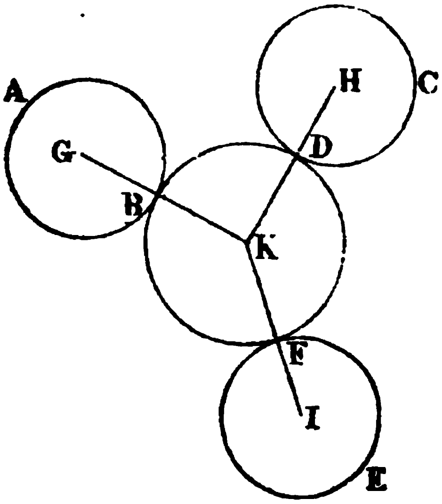
From *L* as a centre, at a distance equal to *M*, describe the circle *CDE*; describe (Supp. xcv. 3.) a circle *CAB*, which shall pass through *A* and *B*, and which shall touch the circle *CDE*; and let *K* be the centre of the circle *CAB*: Then is *K* the point which was to be found.

For join *K*, *L*; therefore, (E. xi. 3. or E. xii. 3.) *KL*, produced, passes through the point *C*, in which the two circles *CDE*, *CAB* touch one another; join, also, *K*, *A* and *K*, *B*; therefore (E. xv. Def. 1.) *KA*, *KB* and *KC* are equal to one another; and $KL = KC - LC$; but (*constr.*) $LC = M$; therefore *KL* is equal to the difference between *KC* and *M*, that is, to the difference between *KA*, or *KB* and *M*.

PROP. XCIX.

128. PROBLEM. *To describe a circle that shall touch three given circles, of which two are equal to one another.*

Let *AB*, *CD*, *EF* be three circles, of which the two *AB* and *CD* are equal to one another: It is required to describe a circle which shall touch the three given circles *AB*, *CD*, and *EF*.



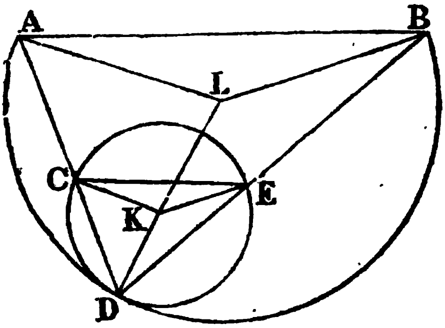
Find (E. i. 3.) the centres *G*, *H* and *I* of the three given circles; find, also, (Supp. xcviii. 3.) a point *K*, the distances of which from *G* and *H*, shall be equal to one another, and shall either of them differ from the distance between the points *K* and *I*, by the semi-diameter of the given circle *EF*; if, then, $\overline{GK}$, $\overline{HK}$ and $\overline{IK}$ be drawn, it is manifest that *KB*, *KD*, and *KF*, are all equal to one another, and, therefore, that a circle, *BDF*, described from the centre *K* at the distance *KB*, will pass through *B*, *D* and *F*, and (Supp. vi. 3.) will touch the circles *AB*, *CD*, and *EF* in the points *B*, *D* and *F*, respectively.

PROP. C.

129. PROBLEM. *To find a point, in the circumference of a given circle, from which if two straight*

lines be drawn to two given points, without the circle, the chord joining the intersections of the lines so drawn and the circumference, shall be parallel to the straight line joining the two given points.

Let *CDE* be a circle, and *A*, *B*, two points with-



out it: It is required to find a point in the circumference of *CDE*, from which if two straight lines be drawn to *A* and *B*, the chord joining their intersections with the circumference of *CDE* shall be parallel to $\overline{AB}$.

Find (E. i. 3.) the centre *K* of the circle *CDE*; find, also, (Supp. xcviii. 3.) a point *L*, the distances of which from *A* and *B*, shall be equal to one another, and shall, either of them, differ from the distance between *L* and *K*, by the semi-diameter of the given circle *CDE*; join *L*, *A* and *L*, *B* and *L*, *K*, and produce *LK* to meet the circumference of *CDE*, in *D*: Then is *D* the point which was to be found.

For (*constr.*) $\overline{LD}$ is equal to $\overline{LA}$ or $\overline{LB}$; and a circle *ADB*, described from *L* as a centre, at the distance *LA*, will (Supp. vi. 3.) touch *CDE* in *D*, and will pass through *B*; draw $\overline{DA}$ and $\overline{DB}$, cutting the circumference of CDE in C and E; join, likewise, C, E, and K, C and K, E: And since (*constr.*) the $\angle CKE$ is an angle at the centre, and the $\angle CDE$ is an angle at the circumference of the circle CDE, the $\angle CKE$ is (E. xx. 3.) the double of the $\angle CDE$; in the same manner, it may be shewn that the $\angle ALB$ is the double of the $\angle ADB$ or CDE; therefore the $\angle CKE = \angle ALB$; therefore (E. xxxii. 1. and E. v. 1.) the $\angle$s KEC, LBA, at the bases of the isosceles Δs CKE, ALB, are equal to one another: Again, since (E. xv. Def. 1.) the Δs EKD, BLD are isosceles, the $\angle KDE = \angle KED$, and the $\angle LDB$ or $KDE = \angle LBD$; therefore the $\angle KED = \angle LBD$; and it has been shewn that the $\angle KEC = \angle LBA$; therefore the whole $\angle CED =$ the whole $\angle ABD$; therefore (E. xxviii. 1.) $\overline{CE}$ is parallel to AB.

A

SUPPLEMENT

TO THE

ELEMENTS OF EUCLID.

PART I.

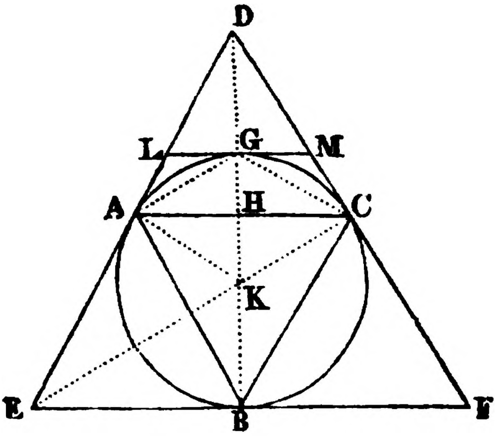
BOOK IV.

PROP. I.

1. THEOREM. *IF an equilateral triangle be described about a given circle, the straight lines joining the points of contact shall contain another equilateral triangle; and the side of the circumscribed triangle is the double of the side of the inscribed triangle so contained.*

Let *ABC* be a circle: About it describe (E. i. 1. and E. iii. 4.) the equilateral △ *DEF*, the sides of which touch the circle in the points *A*, *B* and *C*, respectively; draw $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$: Then is $\overline{ABC}$ an equilateral triangle, and any side, as *EF*, of the △*DEF*, is the double of any side, as *AC*, of the △*ABC*.

For (*constr.* E. v. 1. *Cor.* E. xxxii. 1.) each of the



∠s D, E, F, is the third of two right angles; therefore (Supp. xix. 1. E. v. 1. E. xxxii. 1.) each of the angles of the △s DAC, EAB, FBC, is the third of two right angles, and they are all equal to one another; therefore (E. xxxii. 3.) the △ABC is equiangular; and, therefore, (E. vi. 1. *Cor.*) it is also equilateral.

Again, since it has been shewn that $AB=\overline{AC}$, therefore (E. xxviii. 3.) $\overset{\frown}{AB}=\overset{\frown}{AC}$; therefore (Supp. ix. 3.) $\overline{DE}$ is parallel to $\overline{CB}$; and in the same manner it may be shewn, that $\overline{EF}$ is parallel to AC, and $\overline{DF}$ parallel to AB; therefore the figures $ACBE$; $ACFB$ are parallelograms; therefore (E. xxxiv. 1.) $AC=\overline{EB}$; also $AC=\overline{BF}$; ∴ $EB+BF$, that is EF, is the double of AC.

2. Cor. 1. If K be the centre of the circle, and if K, and any angular point of the circumscribed equilateral triangle, as D, be joined, $\overline{DK}$ is bisected in G, by the arch AGC, and $\overline{KG}$ is bisected in

H, by the side *AC* of the inscribed equilateral triangle.

For draw $\overline{AG}$, $\overline{AK}$, $\overline{CG}$, $\overline{CK}$, and produce *DK* to meet *EF*: Then since (Supp. xix. 3. *Cor.* 1.) $\overline{DA} = \overline{DC}$, and (E. xv. Def. 1.) $AK = CK$, therefore (Supp. i. 3. *Cor.*) *DK* and *AC* bisect one another at right angles in *H*; and the $\angle ADK = \angle CDK$; also $\overline{ED} = \overline{FD}$; ∴ (E. iv. 1.) *DK* produced bisects *EF*, and therefore passes through the point of contact *B*.

Again (E. xxxii. 3.) the $\angle EAB = \angle AGB$ or *AGK*, and (E. v. 1.) the $\angle KAG = \angle AGK$; therefore (E. xxxii. 1.) the angles of the $\triangle AKG$ are equal to the angles of the $\triangle ABE$, which in the proposition was shewn to be equilateral and therefore equiangular; therefore (E. vi. 1. *Cor.*) the $\triangle AKG$ is equilateral; therefore $\overline{AG} = AK$; and in the same manner, it may be shewn that $CG = CK$; ∴ (Supp. i. 3. *Cor.*) $\overline{HG} = \overline{HK}$; and it has been shewn that $\overline{HD} = \overline{HB}$; from these equals take the equals *HG* and *HK*, and there remains $\overline{GD}$ equal to $\overline{KB}$ or $\overline{KG}$; ∴ $\overset{\frown}{AGC}$ bisects *DK* in *G*, and *AC* bisects *KG* in *H*[*].

3. Cor. 2. A straight line which touches a circle, at the extremity of a diameter drawn from the point of contact of any side of an equilateral triangle described about the circle, and which is terminated by the two remaining sides, is the side of an equi-

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[*] From this corollary may be derived an easy practical method of inscribing an equilateral triangle in a given circle, and of describing an equilateral triangle about a given circle.

lateral and equiangular hexagon described about the circle.

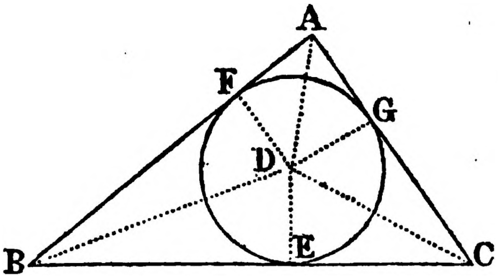
For, from the point B, in which the side EF, of the equilateral $\triangle DEF$, touches the circle ABC, let the diameter BG be drawn, which, as hath been shewn, passes through D, and draw (E. xvii. 1.) $\overline{LM}$ touching the circle in G; draw, also, AG and CG: Then, since it has been proved (*Cor.* 1.) that AG and CG are each of them equal to the semi-diameter of the circle, therefore (E. xv. 4.) they are the sides of an equilateral and equiangular hexagon inscribed in the circle: And if two other tangents be drawn at the extremities of the diameters which pass through the two points A and C, the remaining points of contact may, in the same manner, be shewn to be the remaining angular points of the inscribed hexagon of which AG and GC are sides: And in the same manner as the pentagon described about a circle is proved, in E. xii. 4. to be equilateral and equiangular, may the hexagon thus described about the circle ABC be shewn to be equilateral and equiangular.

4. Cor. 3. An equilateral triangle inscribed in a given circle is a fourth part of the equilateral triangle described about that circle.

Prop. II.

5. Theorem. *If a triangle be described about a given circle, the rectangle contained by the perimeter of the triangle and the semi-diameter of the circle shall be double of the triangle.*

Let *FEG* be a circle: Describe (E. iii. 4.)



any triangle, *ABC*, the sides of which touch the circle in the points *F*, *E*, *G*: The rectangle contained by the semi-diameter of the circle, and the perimeter of the △*ABC*, is double of the △*ABC*.

For take *D* the centre of the circle *FEG*, and draw $\overline{DF}$, $\overline{DG}$, $\overline{DE}$, $\overline{DA}$, $\overline{DB}$ and $\overline{DC}$: Then (E. xli. 1.) the rectangle contained by *DF* and $\overline{AB}$ is double of the △*ADB*, the rectangle contained by *DE* and $\overline{BC}$ is double of the △*BDC*, and the rectangle contained by *DG* and $\overline{AC}$ is double of the △*ADC*; but (E. xv. Def. 1.) *DF*, *DE* and *DG* are equal to one another; if, therefore, *AB*, *BC*, and $\overline{CA}$, be supposed to be placed in the same straight line, the rectangle contained by their aggregate and a semi-diameter of the circle *FEG*, is (E. i. 2.) double of the three △s *ADB*, *BDC*, *CDA*, that is, of the whole △*ABC*.

6. Cor. 1. If any number of triangles with equal perimeters be described about a given circle, they shall be equal to one another.

7. Cor. 2. In the same manner it may be shewn that the rectangle contained by the perimeter of any rectilineal figure, described about a given circle, and the semi-diameter of the circle is double

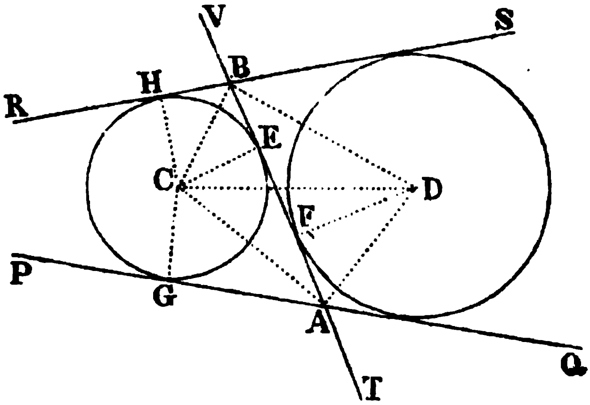
of the rectilineal figure: And, therefore, all rectilineal figures described about the same circle that have equal perimeters, are equal to one another.

PROP. III.

8. PROBLEM. *Three straight lines being given, which, when produced, do not all three meet in the same point, and of which the middle line is not parallel to either of the others, to describe a circle which shall touch each of them.*

Let *PQ, RS, TV*, be three straight lines, which, when produced, do not all meet in the same point: It is required to describe a circle which shall touch $\overline{PQ}$, $\overline{RS}$ and $\overline{TV}$.

Let *PQ* and *RS* cut *TV* in *A*, and *B*; bisect (E. ix. 1.) the $\angle$s *PAB, ABR, QAB, ABS*, by $\overline{AC}$, $\overline{BC}$, $\overline{AD}$ and $\overline{BD}$, and let *AC* and *BC* meet in *C*, and *AD* and *BD* in *D*; from *C* and *D* draw (E. xii. 1.) $\overline{CE}$ and $\overline{DF}$ perpendicular to $\overline{AB}$: Then shall a circle described from the centre *C*, at the



distance CE, touch $\overline{AB}$ and $\overline{AP}$ and $\overline{BR}$; and a circle described from the centre D, at the distance DF, shall touch AB and $\overline{AQ}$ and $\overline{BS}$.

For draw (E. xii. 1.) from C, $\overline{CG}$ perpendicular to $\overline{AP}$, and $\overline{CH}$ perpendicular to $\overline{BR}$, and join C, A and C, B: And because (*constr.*) the $\angle EAC = \angle GAC$, and the angles at E and G are right angles, and that AC is common to the two triangles AEC, AGC, therefore (E. xxvi. 1.) $\overline{CG} = CE$; and in the same manner it may be shewn that $CE = \overline{CH}$; therefore CE, CG, and CH are equal to one another; and therefore a circle described from the centre C at the distance CE will pass through G and H, and (*constr.* and E. xvi. 3. *Cor.*) will touch AB in E, AP in G, and BR in H.

In the same manner it may be proved that a circle described from the centre D, at the distance DF, will touch AB, AQ and BS.

9. Cor. The four points A, C, B and D are in the circumference of a circle.

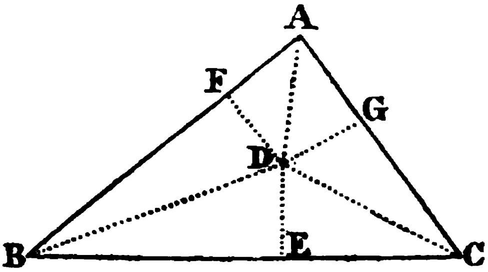
For join C, D: The two $\angle$s CAE, DAE, together, are (*constr.*) the half of the two $\angle$s PAB, QAB taken together; that is the whole $\angle CAD$ is (E. xiii. 1.) the half of two right angles; therefore the $\angle CAD$ is a right angle: In the same manner it may be shewn that the $\angle CBD$ is a right angle; therefore (Supp. xxix. 1. *Cor.* 2.) a circle described upon $\overline{CD}$ as a diameter, will pass through A and B.

Prop. IV.

10. Theorem. *The three straight lines, which*

bisect the three angles of a triangle, meet in the same point.

Let ABC be a triangle: The three straight



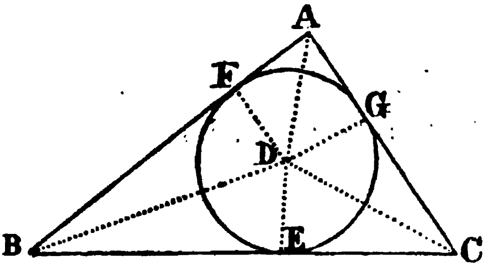
lines which bisect its angles, meet in the same point.

For (E. ix. 1.) bisect the $\angle$s ABC, ACB, by $\overline{BD}$ and $\overline{CD}$, which meet in D, and join A, D; also from D draw (E. xii. 1.) $\overline{DE}$ perpendicular to $\overline{BC}$, $\overline{DF}$ perpendicular to $\overline{AB}$, and $\overline{DG}$ perpendicular to $\overline{AC}$: Then it may be shewn as in the next preceding proposition, that $DF = DG$; and $\overline{DA}$ is common to the two right-angled $\triangle$s AFD, AGD; therefore (Supp. lxxiv. 1.) the $\angle FAD = \angle GAD$; $\therefore$ $\overline{AD}$ bisects the $\angle BAC$, and the three straight lines which bisect the three angles of the $\triangle ABC$ meet in the same point D.

PROP. V.

11. THEOREM. *If a circle be inscribed in a right-angled triangle, the excess of the two sides, containing the right angle, above the third side, is equal to the diameter of the inscribed circle.*

Let ABC be a triangle having one of its $\angle$s BAC, a right angle; and let (E. iv. 2.) the circle *FEG*, of



which *D* is the centre, be inscribed in it. The excess of $AB+AC$ above *BC* is equal to the diameter of the circle *FEG*.

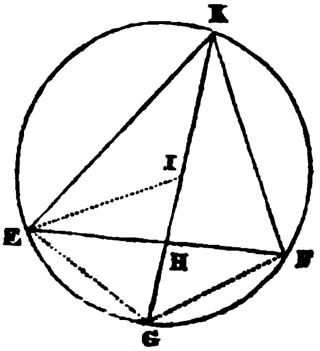
For join the centre *D*, and the points of contact *E*, *F*, and *G*; join, also, *D*, *A*: Then since (Supp. xix. 3. *Cor.* 1.) $\overline{BE}=\overline{BF}$, and $\overline{CE}=\overline{CG}$, it is evident that $\overline{AF}+\overline{AG}$, or $2\overline{AF}$, is the excess of $\overline{AB}+\overline{AC}$ above $\overline{BC}$: Again, since $AF=AG$, and $\overline{FD}=\overline{GD}$, and $\overline{AD}$ is common to the two $\triangle$s *AFD*, *AGD*, therefore (E. viii. 1.) the $\angle FAD=\angle GAD$; but (*hyp.*) the $\angle FAG$ is a right angle; therefore the $\angle FAD$ is half of a right angle; also (*constr.* and E. xviii. 3.) the $\angle AFD$ is a right angle; therefore (E. xxxii. 1.) the $\angle FDA$ is half of a right angle; therefore the $\angle FAD=\angle FDA$; therefore (E. vi. 1.) $AF=FD$, a semi-diameter of the circle *FEG*; $\therefore$ $2AF$, which was shewn to be the excess of $AB+AC$ above *BC*, is equal to the diameter of *FEG*.

PROP. VI.

12. THEOREM. *The straight line bisecting any angle of a triangle, inscribed in a given circle, cuts the circumference, in a point which is equidistant*

from the extremities of the side opposite to the bisected angle, and from the centre of a circle inscribed in the triangle.

Let *KEF* be a triangle inscribed in the circle *KEGF*, and let $\overline{KG}$, which bisects the $\angle EKF$, meet the circumference in *G*: The point *G* is equidistant from *E* and *F*, and from the centre of the circle inscribed in the $\triangle KEF$.

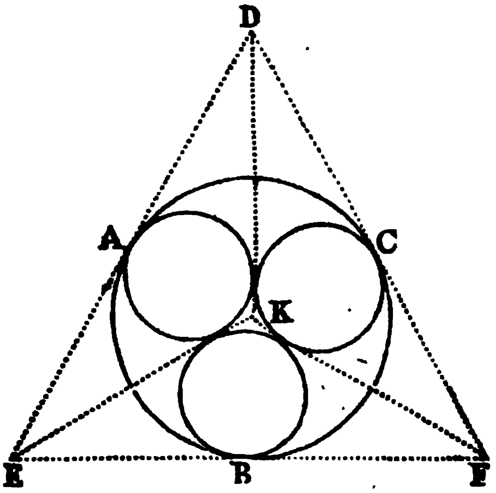


For, join *G*, *E*, and *G*, *F*; draw (E. ix. 1.) $\overline{EI}$ bisecting the $\angle KEF$; therefore (E. iv. 4.) *I* is the centre of the circle inscribed in the $\triangle KEF$: And since (*hyp.*) the $\angle EKG = \angle FKG$, therefore (E. xxvi. 3.) $\overset{\frown}{GE} = \overset{\frown}{GF}$, and therefore (E. xxix. 3.) $\overline{GE} = \overline{GF}$: Again, because (E. xxi. 3.) the $\angle GEF = \angle GKF$, therefore (*constr.*) the two $\angle$s *GEH*, *HEI*, that is, the $\angle GEI$, are equal to half of the two $\angle$s *EKF*, *FEK*; also the exterior $\angle EIG$, of the $\angle EIK$, is (E. xxxii. 1.) equal to the two $\angle$s *IKE*, *KEI*, that is (*constr.*) to half of the two $\angle$s *FKE*, *KEF*; therefore the $\angle EIG = \angle GEI$; therefore (E. vi. 1.) $GE = \overline{GI}$; and it has been shewn that $GE = GF$; therefore *G* is equidistant from *E*, and *F*, and from the centre *I* of the circle inscribed in the $\triangle KEF$.

PROP. VII.

13. PROBLEM. *In a given circle, to inscribe three equal circles, touching each other and the given circle.*

Let *ABC* be the given circle: It is required to



inscribe it in three equal circles, touching each other, and also touching *ABC*.

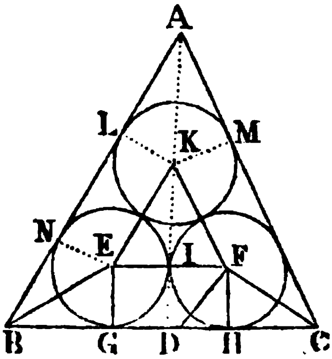
About the circle *ABC* describe (E. i. 1. and E. iii. 4.) the equilateral △*DEF*, the sides of which touch the circle in the points *A*, *B* and *C*: Take the centre *K*, and draw $\overline{KD}$, $\overline{KE}$ and $\overline{KF}$: It is manifest, from the demonstration of E. iv. 4., and of Supp. i. 4. *Cor.* 1., that, if a circle be inscribed in each of the equal △s *DKE*, *EKF*, *FKD*, these inscribed circles will be equal, and will touch one another.

PROP. VIII.

14. PROBLEM. *To inscribe three circles in an equilateral triangle, touching each other, and each*

of them touching two of the three sides of the triangle.

Let *ABC* be an equilateral triangle. It is required to inscribe in the △*ABC,* three circles that touch one another, and two sides of the triangle.



Bisect (E. ix. 1.) the vertical ∠*BAC* by $\overline{AD}$; therefore (E. iv. 1.) the two △s *ADB,* *ADC,* are right-angled at *D*; in the △*ADB* inscribe (E. iv. 4.) the circle *IG*; and in the △*ADC* inscribe the circle *IH*: Then, it is manifest, from the demonstration of E. iv. 4, that the two circles *IG* and *IH* touch *AD* in the same point *I*, and therefore touch one another in that point, and that they are equal to one another; take their centres *E* and *F*, and draw $\overline{EF}$, which (E. xii. 3.) passes through *I*; and since the two circles are equal, *EF* is bisected in *I*; join, also, *E*, *G*, and *F*, *H*; $\overline{EG}$ is (E. xviii. 3. and E. xxviii. 1.) parallel to $\overline{FH}$; and $\overline{EG}=\overline{FH}$, therefore (E. xxxiii. 1.) *EF* is parallel to *BC*, and (E. xxix. 1.) the ∠s *AIE*, *AIF*, are right angles; if, therefore, from the centre *E*, at the distance *EF*, a circle be described, cutting *AI* in *K*, and if *K*, *F* be joined, $\overline{KF}$

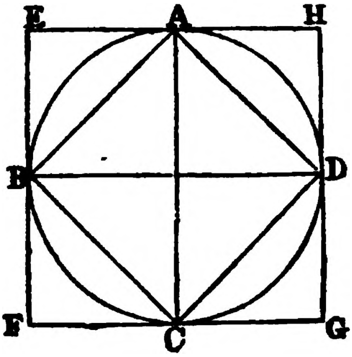
(E. iv. 1.) = $\overline{KE}$, and the △*KEF* is equilateral; and its ∠*EKF*, which (E. xxxii. 1.) is equal to the ∠*BAC*, is bisected by $\overline{AKI}$; therefore the ∠*EKI* = ∠*BAD*; therefore (E. xxviii. 1.) *KE* is parallel to *AB*; join *E*, *N*, and draw (E. xii. 1.) $\overline{KL}$ perpendicular to $\overline{AB}$, and therefore (E. xxviii. 1.) parallel to *EN*; ∴ *KLNE* is a parallelogram, and (E. xxxiv. 1.) *KL* = $\overline{EN}$, or $\overline{EI}$, or the half of $\overline{EK}$; and if *KM* be drawn perpendicular to $\overline{AC}$, it is equal (*constr.* and E. xxvi. 1.) to *KL*. It is evident, therefore, that a circle, *LM*, described from the centre *K*, at the distance *KL*, or *KM*, will (E. xvii. 3. *Cor.* and Supp. vi. 3.) touch *AB* and $\overline{AC}$, and each of the circles *GI*, and *HI*: And thus will three circles have been inscribed in the equilateral △*ABC* touching one another, and each of them touching two sides of the triangle.

PROP. IX.

15. THEOREM. *The square, inscribed in a circle, is equal to the half of the square upon its diameter.*

Let *ABCD* be a circle: Inscribe in it (E. vi. 4.) by drawing the diameters *AC*, and *BD* perpendicular to one another, the square *ABCD*, and describe about it (E. vii. 4.) the square *EFGH*: And since (E. xli. 1.) the △*BAD* is half of the ▭*EBDH*, and the △*BCD* is half of the ▭ *BFGD*, the two △s *BAD*, *BCD* are, together, half of the two ▭s *EBDH*, *BFGD*; that is, the inscribed square

ABCD is half of the circumscribed square *EFGH*,

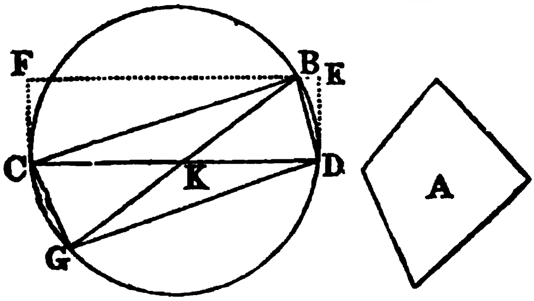


which is equal to the square upon the diameter, because (E. xxxiv. 1.) its side *FG* equal the diameter *BD* of the circle.

Prop. X.

16. Problem. *In a given circle, to inscribe a rectangle equal to a given rectilineal figure, not exceeding the half of the square upon the diameter.*

Let *A* be the given rectilineal figure, and *BCD* the given circle: It is required to inscribe, in the circle *BCD*, a rectangle equal to the figure *A*.



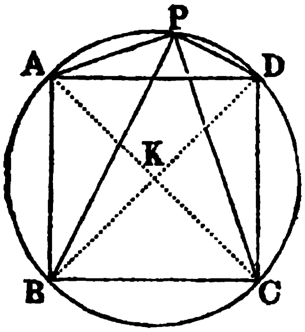
Draw any diameter *CD* of the given circle; find (Supp. iv. 1. *Cor.*) a triangle equal to *A*; and to

CD apply (E. xliv. 1.) a ▭*CDEF*, equal to that triangle, and, therefore, equal to the given figure *A*; let the side *EF* of the ▭*CDEF* cut the circumference of the given circle *CBD* in *B*; draw the diameter *BKG*, and join *C*, *B*, and *B*, *D*, and *C*, *G* and *D*, *G*: And, since (E. xxxi. 3.) each of the ∠s *CBD*, *BDG*, *DGC*, *GCB*, are right angles, therefore (Supp. xxxvi. 1.) *BDGC* is a rectangular parallelogram; and therefore (E. xxxiv. 1.) it is double of the △*CBD*; also (E. xli. 1.) the ▭*EDCF* is double of the △*CBD*; therefore the rectangle *CBDG*=▭*EFCD*, which has been shewn to be equal to *A*; therefore the rectangle *CBDG*=*A*.

PROP. XI.

17. THEOREM. *If from any point, in the circumference of a given circle, straight lines be drawn to the four angular points of an inscribed square, the aggregate of the squares of the four lines, so drawn, shall be the double of the square of the diameter.*

Let *ABCD* be a circle; inscribe in it (E. vi. 4.)



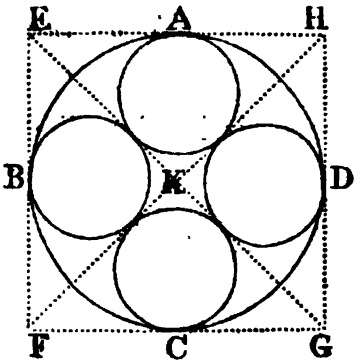
the square *BADC*, and from any point *P*, in the circumference, let there be drawn to the angular points A, B, C, D, $\overline{PA}$, $\overline{PB}$, $\overline{PC}$ and $\overline{PD}$; Then $PA^2+PB^2+PC^2+PD^2$ shall be double of the square of the diameter.

For let K be the centre of the circle, and $\overline{AKC}$, $\overline{DKB}$ the two diameters perpendicular to one another, by joining the extremities of which (E. vi. 4.) the square was inscribed in the circle: Then since (E. xxxi. 3.) the ∠s APC, BPD are right angles, therefore (E. xlvii. 1.) $PA^2+PC^2=\overline{AC}^2$, and $PB^2+PD^2=\overline{DB}^2$ or AC^2; $\therefore PA^2+PB^2+PC^2+PD^2=2AC^2$.

Prop. XII.

18. Problem. *In a given circle, to inscribe four circles equal to each other, and in mutual contact with each other and the given circle.*

Let $ABCD$ be the given circle: It is required



to inscribe in it four equal circles touching one another, and the circle $ABCD$.

About the circle $ABCD$ describe (E. vii. 4.) the square $EFGH$, and draw its diagonals EG, HF, which cut one another in the centre K, so that

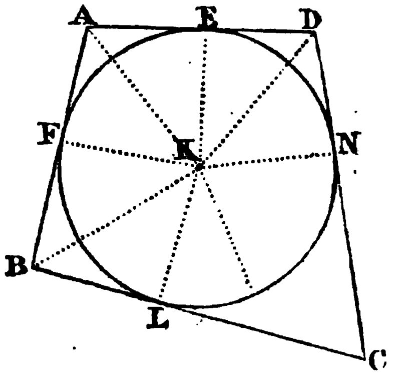
(E. xxvi. 1.) the four △s *EKF*, *FKG*, *GKH*, *HKE*, have their sides and angles respectively equal to one another: It is manifest, therefore, from the demonstration of E. iv. 4., that if a circle be inscribed in each of the four equal triangles, the circles so described, will be equal, and will touch one another; and the given circle *ABCD*.

19. Cor. In the same manner, four equal circles may be inscribed in a given square, touching each other and the sides of the square.

Prop. XIII.

20. Problem. *To inscribe a circle in a given trapezium, of which two opposite sides are, together, equal to the other two sides taken together.*

Let *ABCD* be the given trapezium, having the two sides *AD* and *BC* equal, together, to the two remaining opposite sides *AB* and *DC*: It is required to inscribe a circle in the trapezium *ABCD*.



Bisect (E. ix. 1.) each of the ∠s *BAD*, *ADC*, by $\overline{AK}$ and $\overline{DK}$, which meet in *K*; from *K* draw (E. xii. 1.) $\overline{KE}$ perpendicular to $\overline{AD}$, $\overline{KF}$ perpendicular to $\overline{AB}$, $\overline{KL}$ perpendicular to $\overline{BC}$, and $\overline{KN}$ perpendicular to $\overline{CD}$: Then (*demonstr.* of Supp. iii. 4.) KE, KF, and KN are equal to one another; as are, also, $\overline{AF}$ and $\overline{AE}$, and $\overline{DE}$ and $\overline{DN}$; and KL is equal to KF or KN: For if KL be not equal to KF or KN, it is either greater or less; if it be possible, let $KL > KF$ or KN; and join K, A, and K, D, and K, C, and K, B: Then (*constr.* and E. xlvii. 1.) $\overline{KB}^2 = KF^2 + \overline{BF}^2$; and, likewise, $KB^2 = KL^2 + \overline{BL}^2$ $\therefore KF^2 + BF^2 = KL^2 + BL^2$; but $KL^2 > KF^2$; $\therefore BL^2 < BF^2$, and $BL < BF$: In the same manner it may be shewn that $\overline{LC} < \overline{CN}$; $\therefore BL + LC$, or $\overline{BC}$, $< BF + CN$; add $\overline{AD}$ to BC, and $\overline{AF} + \overline{DN}$, which $= AD$, to $BF + CN$; $\therefore AD + BC < \overline{AB} + \overline{DC}$, which is contrary to the supposition; $\therefore KL$ is not $> KF$; and in a similar manner it may be shewn that KL is not $< KF$; $\therefore KL = KF$, or KN, or KE: From K, therefore, as a centre, at the distance KF, describe a circle $EFLN$, and it will pass through the points L, G, and E, and (E. xvi. 3. *Cor.*) will touch AB, BC, $\overline{CD}$ and $\overline{DA}$, respectively, in the points F, L, N, and E.

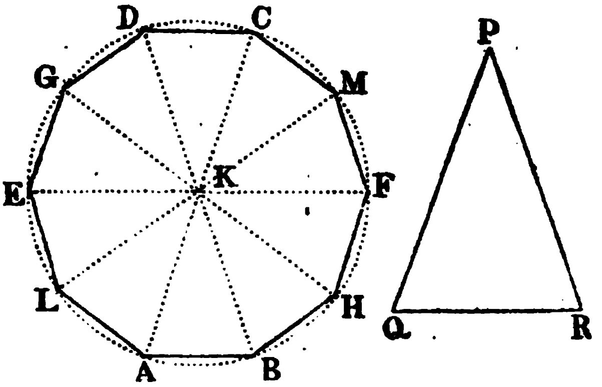
21. Cor. If two opposite sides of a trapezium be together equal to the other two sides, taken together, the four straight lines, which bisect the four angles of the figure, all of them meet in the same point.

Prop. XIV.

22. Problem. *Upon a given finite straight line, to describe an equilateral and equiangular decagon.*

Let AB be the given straight line: It is required to describe upon it an equilateral and equiangular decagon.

Describe (E. x. 4.) the △ *PQR*, having each of



the ∠s *PQR*, *PRQ*, double of the ∠ *P*; at the points *A* and *B*, in $\overline{AB}$, make (E. xxiii. 1.) the ∠s *BAK*, *ABK*, each of them equal to the ∠ *PQR*, or *PRQ*; ∴ (Supp. xxvi. 1.) the ∠ *AKB* = ∠ *QPR*, and the ∠s *KAB*, *KBA*, are each of them the double of the ∠ *AKB*, which angle is, therefore, the fifth part of two right angles; from the centre *K*, at the distance *KA* or *KB*, describe the circle *ABCD*, cutting $\overline{AK}$ and $\overline{BK}$, produced, in *C* and *D*; bisect (E. ix. 1.) the ∠ *DKA* by $\overline{EKF}$, which (E. xv. 1.) also bisects the ∠ *CKB*; again bisect the ∠s *DKE*, *EKA*, by $\overline{GKH}$, $\overline{LKM}$, which also bisect the angles *FKB*, *CKF*; lastly, draw $\overline{BH}$, $\overline{HF}$, $\overline{FM}$, $\overline{MC}$, $\overline{CD}$, $\overline{DG}$, $\overline{GE}$, $\overline{EL}$ and $\overline{LA}$: The ten-sided figure *ABHFMCDGEL* is an equilateral and equiangular decagon.

For the ∠ *AKB* has been shewn to be the fifth part of two right angles; therefore (E. xiii. 1.) it is the fifth part of the ∠s *AKB*, *AKD*; therefore the ∠ *AKD* = 4 ∠ *AKB*; therefore (*constr.*) the angles *BKA*, *AKL*, *LKE*, *EKG*, *GKD*, and (E. xv. 1.)

their vertical angles are equal to one another; therefore (E. xxvi. 3. and E. xxix. 3.) the ten-sided figure is equilateral; and since (*constr.* and E. xxxii. 1.) the isosceles triangles, into which it is divided by the straight lines drawn from *K* to its angular points, have the angles at their bases all equal, the figure is also equiangular.

23. Cor. 1. It is manifest from this proposition, and from E. x. 4., that if the semi-diameter of a circle be divided into two parts, so that the rectangle contained by the whole and the lesser part may be equal to the square of the greater part, the greater segment shall be equal to the side of an equilateral and equiangular decagon inscribed in the circle; and thus may such a decagon be inscribed in a given circle.

24. Cor. 2. In the solution of the proposition, is shewn the method of describing, upon a given finite straight line as a base, an isosceles triangle, having each of the angles at the base double of the third angle.

25. Cor. 3. The figure *ABHFMCDGEL* being an equilateral and equiangular decagon, if the points *A, H,* and *H, M,* and *M, D,* and *D, E,* and *E, A,* be joined, it may be shewn, from E. iv. 1., that the figure *AHMDE* is an equilateral and equiangular pentagon.

In the same manner, if an equilateral and equiangular rectilineal figure of any *even* number of sides be given, such a figure, having half that number of sides, may be constructed: Also, if a circle be described about the given figure, which can always be done by the method used in E. xiv. 4., and, each of the equal angles of the figure having (E. ix. 1.) been bisected, if the points in which the circumference is met by the bisecting lines, and the angular points of the given figure be joined, a figure of twice as many sides as the given figure will have been constructed, which (E. xxvi. 3., and E. xxix. 3.), is equilateral, and, therefore, equiangular: For in the same manner, that an equilateral pentagon, inscribed in a circle, is shewn (E. xi. 2.) to be equiangular, may any other equilateral rectilineal figure, inscribed in a circle, be shewn to be equiangular.

Thus, by the help of E. ix. 1, E. ii. 4, E. vi. 4, E. xi. 4, and E. xvi. 4., equilateral and equiangular figures may be inscribed in a given circle, of three, six, twelve, &c., equal sides; of four, eight, sixteen, &c. equal sides; of five, ten, twenty, &c. equal sides; and of fifteen, thirty, sixty, &c. equal sides.

PROP. XV.

26. PROBLEM. *Upon a given finite straight line, to describe an equilateral and equiangular pentagon.*

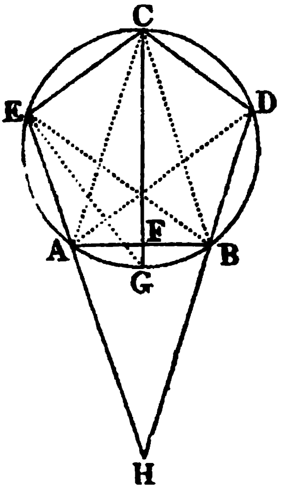
If upon the given finite straight line an isosceles triangle be described (Supp. xiv. 4. *Cor.* 2.) having each of the angles at the base double of the third angle, and if, also, a circle be described (E. v. 4.) about that triangle, it will be manifest, that the equilateral and equiangular pentagon inscribed in the circle, according to the method used in E. xi. 4., is the figure which was to be constructed.

PROP. XVI.

27. THEOREM. *The angle of a regular pentagon exceeds a right angle by one-fifth part of a right*

angle; and is three times as great as the angle contained by any two sides of the figure, which are not adjacent to each other, produced so as to meet.

Let $ABDCE$ be the given equilateral and equiangular pentagon, and let any two of its sides, as EA, DB, be produced, so as to meet in H: Any of its angles exceeds a right angle by one-fifth part of a right angle, and is three times as great as the angle AHB.



About the pentagon $ABDCE$ describe (E. xiv. 4.) the circle $AECDB$; bisect (E. xxx. 3.) $\widehat{AB}$, in G, and join C, A and C, B, and C, G and E, G and E, B: And since (*hyp*. and E. xxviii. 3.) the circumferences $\widehat{CE}$, $\widehat{EA}$, $\widehat{CD}$, $\widehat{DB}$, are equal, $\widehat{CEG}=\widehat{CDG}$; $\therefore$ $\widehat{CEG}$ is the semi-circumference of the circle, and (E. xxxi. 3.) the $\angle CEG$ is a right angle; also (E. xxi. 3.) the $\angle AEG = \angle ACG$; and it is manifest from the demonstration of E. xi. 4. E. xxxii. 1., that the $\angle ACB$ is the fifth part of two right angles; and therefore that its half, namely the $\angle\, AEG$, is the fifth part of a right angle; therefore the $\angle\, AEG$, which is the excess of the $\angle\, CEA$ above a right angle, is the fifth part of a right angle.

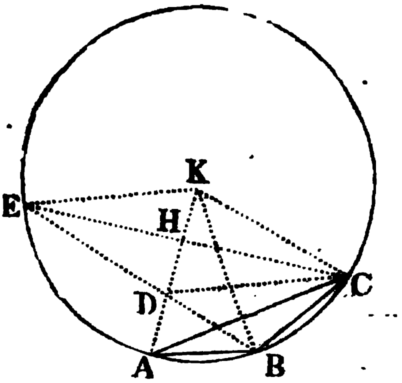
Again, the two opposite $\angle$s AEC, CBA, of the trapezium $AECB$ are (E. xxii. 3.) together equal to two right angles; and (E. xxvii. 3.) the $\angle\, CBA = \angle\, BCE$; therefore the $\angle\, AEC + \angle\, ECB =$ two right angles, and therefore (E. xxviii. 1.) $\overline{CB}$ is parallel to $\overline{EA}$ or $\overline{EH}$; therefore (E. xxix. 3.) the $\angle\, EHD = \angle\, CBD$, which, since (E. xxvii. 3.) the three $\angle$s CBD, CBE, and EBA, are equal to one another, is a third part of the $\angle\, ABD$ of the pentagon $ABDCE$.

28. Cor. It is manifest from the demonstration, that the straight line joining the extremities of the first and second side of an equilateral and equiangular pentagon is parallel to the fourth side of the figure; the sides being taken in order from any one of them assumed as the first.

Prop. XVII.

29. Theorem. *The square of the side of a regular pentagon, inscribed in a given circle, is equal to the square of the side of a regular decagon, together with the square of the side of the regular hexagon, both inscribed in that given circle.*

Let $AECB$ be the given circle, of which K is the centre, and $\overline{AB}$ the side of a regular decagon inscribed (Supp. xiv. 4. *Cor.* 1.) in it: Place, in the circle, $\overline{BC} = AB$, and join A, C; therefore (Supp. xiv. 4. *Cor.* 3.) $\overline{AC}$ is the side of a regular pentagon inscribed in the circle, and if $\overline{KA}$, $\overline{KB}$, and $\overline{KC}$ be drawn, any one of these lines, as KA, is (E. xv. 4.)



the side of a regular hexagon inscribed in the circle: Then $AC^2=KA^2+AB^2$.

For, in KA take $\overline{KD}=AB$, draw $\overline{BD}$ and produce it to meet the circumference in E; also, draw $\overline{KE}$, KC, $\overline{CE}$ and $\overline{CD}$: Then, it is manifest from Supp. xiv. 4. *Cor.* 1. and E. x. 4., that the $\angle KBD = \angle AKB$; but (*constr.* and E. viii. 1.) the $\angle AKB = \angle BKC$; therefore the $\angle DBK$ or $EBK = \angle BKC$; therefore (E. xxvii. 1.) $\overline{ED}$ is parallel to KC: Again, (E. xx. 3.) the $\angle EKA$ or $EKD = 2\angle EBA$; but (E. xxxii. 1.) the exterior $\angle EDK$ is equal to the $\angle DKB + \angle KBD$, that is, (*constr.*) to twice the $\angle KBE$, or to twice the $\angle EBA$; therefore the $\angle EKD = \angle EDK$; ∴ $ED = \overline{EK}$; and (E. xv. Def. 1.) $EK = KC$; ∴ $ED = KC$, and it has been shewn that ED is parallel to KC; therefore (E. xxxiii. 1.) EK is equal and parallel to $\overline{DC}$, and the figure $EKCD$ is a rhombus; therefore (Supp. xlv. 1.) KD is bisected at right angles in H, by $\overline{CE}$: And since KD is bisected in H and produced to A, therefore (E. vi. 2.)

$$KA \times \overline{AD} + \overline{DH}^2 = \overline{AH}^2;$$

$$\therefore\ KA \times AD + DH^2 + \overline{HC}^2 = AH^2 + HC^2;$$

but (*constr.* and Supp. xiv. 4. *Cor.* 1.)

$$KA \times AD = AB^2;$$

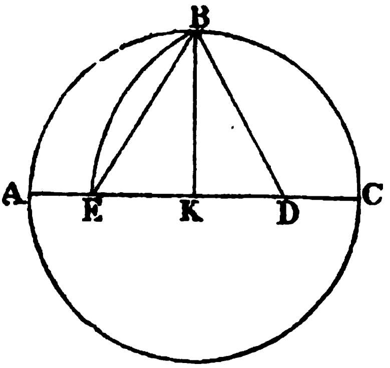
and (E. xlvii. 1.)

$$DH^2 + HC^2 = DC^2, \text{ or } KC^2, \text{ or } KA^2;$$

$$\text{and } AH^2 + \overline{HC}^2 = AC^2;$$

$$\therefore\ AC^2 = KA^2 + AB^2.$$

30. COR. Hence, if *ABC* be a given circle,



and $\overline{AC}$, $\overline{BK}$ two diameters drawn (E. xi. 1.) at right angles to one another, if *KC* be (E. x. 1.) bisected in *D*, and if from $\overline{DA}$ there be cut off $\overline{DE} = \overline{DB}$, $\overline{EK}$ is the side of a regular decagon inscribed in the circle *ABC*, *KB* is the side of a regular hexagon inscribed in it, and $\overline{EB}$ is the side of a regular pentagon inscribed in it.

For (E. xi. 2.) *EK* is equal to that part of *KB*, the square of which equals the rectangle contained by *KB* and the remaining part of *KB*; therefore (Supp. xiv. 4. *Cor.* 1.) *EK* is the side of a regular decagon; and *KB* (E. xv. 4.) is the side of the regular hexagon, inscribed in the circle *ABC*; since, therefore, (*constr.* and E. xlvii. 1.) $EB^2 = EK^2 + KD^2$, EB is (Supp. xvii. 4.) the side of a regular pentagon inscribed in the circle ABC*.

PROP. XVIII.

31. PROBLEM. *Upon a given finite straight line, to describe an equilateral and equiangular hexagon.*

Upon the given straight line, as a base, describe (E. i. 1.) an equilateral triangle; from its vertex as a centre, at the distance of either of its sides, describe a circle: Then, if a regular hexagon be inscribed in the circle by the method used in E. xv. 4., taking either extremity of the base of the equilateral triangle, for the centre of the circle to be next described, it is manifest that the given straight line will be one of its sides.

PROP. XIX.

32. PROBLEM. *A circle being given, to describe six other circles, each of them equal to it, and in contact with each other and with the given circle.*

Let IGH be the given circle: It is required to describe six other circles, equal each of them to the circle IGH, and touching that circle and each other.

Find (E. i. 3.) the centre K, of the circle IGH; take any of its semi-diameters, as $\overline{KG}$; produce KG

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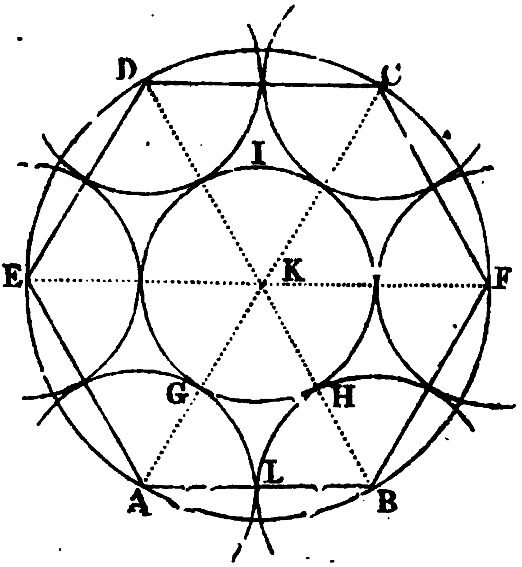
* This corollary furnishes the best practical method of determining the sides of a regular pentagon, and of a regular decagon, to be inscribed in a given circle; and thus makes it easier to describe a regular pentagon, or a regular decagon, on a given finite straight line.

to *A*, and make $\overline{GA}=\overline{GK}$; from the centre *K*, at the distance *KA*, describe the circle *ABCD*; and in the circle *ABCD* inscribe (E. xv. 4.) the equilateral and equiangular hexagon *ABFCDE*; bisect (E. x. 1.) the side *AB* of the hexagon in *L*: Then since (E. xv. 4.) $\overline{AB}=\overline{KA}$ or $\overline{KB}$, and (*constr.*) $\overline{AG}$ and $\overline{BH}$ are, each of them, the half of *KA* or *KB*; ∴ *AG* and *BH* are each of them equal to $\overline{AL}$ or $\overline{BL}$: If, therefore, from the centres *A* and *B*, at the distance *AG*, or *BH*, two circles be described, they will be equal to one another and to the given circle, and they will touch (Supp. vi. 3.) the given circle in *G* and *H*, and will, also, touch one another in *L*. In the same manner, from the points *E*, *D*, *C*, *F*, as centres, may four other circles be described, each equal to the given circle and in contact with it, and touching also each other.



33. Cor. No more than six circles can be described touching one another and a given circle, and each of them equal to the given circle.

PROP. XX.

34. PROBLEM. *In a given circle to inscribe six circles equal to one another, touching, each of them, the given circle, and touching, also, one another.*

Inscribe (E. xv. 4.) an equilateral and equiangular hexagon in the given circle, and through the points, in which are its angles, draw, (E. xvii. 3.) straight lines touching the circle; and it may be shewn, by the method used in E. xii. 4., that the figure contained by these tangents is an equilateral and equiangular hexagon; from the centre of the circle draw straight lines to the several angles of the circumscribed hexagon, thus dividing it into six equal equilateral triangles; and if (E. iv. 4.) a circle be inscribed in each of these triangles, it will be manifest from the demonstration of E. iv. 4., that the circles, so inscribed, will be equal, and that they will touch one another in common points of the sides of the triangles, and will, also, touch the given circle, each of them in one of the points of contact of the circumscribed hexagon.

A

SUPPLEMENT

TO THE

ELEMENTS OF EUCLID.

PART I.

BOOK V.

PROP. I.

1. **THEOREM.** *If the first of four proportional magnitudes be greater than the second, the third is also greater than the fourth; if equal, equal; and if less, less.*

Let $A : B :: C : D$; and first, let $A > B$; then $C > D$.

Take the doubles of the four magnitudes; and since (*hyp.*) $A > B$, twice $A >$ twice B; therefore (*hyp.* and v. Def. 5.) twice $C >$ twice D; $\therefore C > D$.

In like manner it can be shewn, if $A = B$, that $C = D$; and if $A < B$, that $C < D$.

PROP. II.

2. **THEOREM.** *If four magnitudes are proportionals, they are proportionals also when taken inversely.*

Let $A : B :: C : D$; then $B : A :: D : C$.

Let there be taken of A and C any equi-multiples pA, pC, and of B and D any equi-multiples qB, qD: Then (*hyp.*) $A : B :: C : D$; therefore (v. Def. 5.) if $pA > qB$, $pC > qD$; if $pA = qB$, $pC = qD$; if $pA < qB$, $pC < qD$; therefore accordingly as qB is greater than, equal to, or less than pA, qD is greater than, equal to, or less than pC;

therefore (v. Def. 5.) $B : A :: D : C$.

PROP. III.

3. THEOREM. *If the first of four magnitudes be the same multiple of the second, or the same part of it, that the third is of the fourth, the first is to the second, as the third is to the fourth.*

First, let $A = pB$, and $C = pD$;

then $A : B :: C : D$.

Let there be taken of A and C any equi-multiples qA, qC, and of B and D any equi-multiples rB, rD. And since $A = pB$, accordingly as qA is greater, equal to, or less than rB, will q times pB be greater, equal to, or less than rB, that is, q times p will be greater, equal to, or less than r; and therefore q times pD will also be greater, equal to, or less than rD; that is, (*hyp.*) qC will be greater, equal to, or less than rD; therefore (v. Def. 5.)

$A : B :: C : D$.

Secondly, let $pA = B$, and $pC = D$; then, also,

$A : B :: C : D$.

For in that case, as hath been shewn,

$B : A :: D : C$;

therefore (Supp. ii. 5.)

$A : B :: C : D$.

PROP. IV.

4. THEOREM. *If the first of four proportional magnitudes be a multiple, or a part, of the second, the third is the same multiple, or the same part, of the fourth.*

If $A : B :: C : D$, and if $A=pB$, then $C=pD$.

For (*hyp.* and Supp. iii. 5.) $A : B :: pD : D$;
and (*hyp.*) $A : B :: C : D$;
therefore (E. xi. 5.) $C : D :: pD : D$;
therefore (E. ix. 5.) $C=pD$.

Again, if $A : B :: C : D$, and if $pA=B$, then $pC=D$.

For (*hyp.*) $A : B :: C : D$;
$\therefore$ (Supp. ii. 5.) $B : A :: D : C$;

and (*hyp.*) $B=pA$; therefore, as in the former case, $D=pC$; that is, C is the same part of D, that A is of B.

PROP. V.

5. THEOREM. *If any number of equal ratios be each greater than a given ratio, the ratio of the sum of their antecedents to the sum of their consequents, shall be greater than that given ratio.*

Let the ratios $(A : B)$, $(C : D)$, $(E : F)$, &c. be equal to one another, and let each of them be greater than the ratio $(P : Q)$; then $(A+C+E : B+D+F) > (P : Q.)$

For (E. xii. 5.) $A+C+E : B+D+F :: A : B$;
and (*hyp.*) $(A : B) > (P : Q)$;
$\therefore (A+C+E : B+D+F) > P : Q$.

PROP. VI.

6. THEOREM. *If the first of four magnitudes have*

a greater ratio to the second than the third has to the fourth, the second shall have to the first a less ratio than the fourth has to the third.

If $(A : B) > (C : D)$, then is $(B : A) < (D : C)$.

For, let E be a magnitude such that

$$(E : B) :: (C : D);$$

and since (*hyp.*)

$$(A : B) > (C : D); \therefore (A : B) > (E : B);$$

therefore (E. x. 5.) $E < A$;

therefore (E. viii. 5.) $(B : E) > (B : A)$:

But (*hyp.* and Supp. ii. 5.) $(D : C) :: (B : E)$;

therefore (E. xiii. 5.) $(D : C) > (B : A)$:

Or, $(B : A) < (D : C)$.

PROP. VII.

7. THEOREM. *If the first of four magnitudes, of the same kind, have a greater ratio to the second than the third has to the fourth, the first shall have to the third a greater ratio than the second has to the fourth.*

If $(A : B)$ be greater than $(C : D)$, then is

$$(A : C) > (B : D).$$

For, let E be a magnitude such that

$$(E : B) :: (C : D);$$

therefore (*hyp.* and E. x. 5.) $A > E$;

therefore (E. viii. 5.) $(A : C) > (E : C)$;

But (E. xvi. 5. and *hyp.*) $(E : C) :: (B : D)$;

$$\therefore (A : C) > (B : D).$$

PROP. VIII.

8. THEOREM. *If four magnitudes of the same kind be proportionals, and if the first of them be the*

greatest, the fourth shall be the least; but if the first of them be the least, the fourth shall be the greatest.

Let A, B, C, D, be four magnitudes of the same kind, which are proportionals; and, first, let A be the greatest; then D shall be the least of them.

For, since (*hyp.*) $A > C$; therefore (E. xiv. 5.) $B > D$;

Again, since (*hyp.*) $A : B :: C : D$;
therefore (E. xvi. 5.) $A : C :: B : D$:

But (*hyp.*) $A > B$; therefore (E. xiv. 5.) $C > D$: And it has been shewn that $B > D$; therefore D is in this case the least of the four proportionals. And, if A be the least of the four proportionals, it may, in like manner, be proved that D will be the greatest of them.

9. Cor. If four magnitudes of the same kind, be continual proportionals, the difference between the two extremes is greater than the difference between the two means.

Prop. IX.

10. Theorem. *If the first, together with the second, of four magnitudes, have a greater ratio to the second, than the third, together with the fourth, has to the fourth, the first shall have a greater ratio to the second than the third has to the fourth.*

If $(A+B : B) > (C+D : D)$,
then is $(A : B) > (C : D)$.

For, let E be a magnitude such that
$(E+B : B) :: (C+D : D)$;
therefore (E. x. 5.) $A+B > E+B$;

$$\therefore A > E;$$

therefore (E. viii. 5.) $(A : B) > (E : B)$:

But (*hyp*. and E. xvii. 5.) $(E : B) = (C : D)$;

$$\therefore (A : B) > (C : D).$$

PROP. X.

11. THEOREM. *If the first of four magnitudes have a greater ratio to the second than the third has to the fourth, the first, together with the second, shall have to the second, a greater ratio than the third, together with the fourth, has to the fourth.*

If $(A : B) > (C : D)$;

then is $(A+B : B) > (C+D : D)$.

For, let E be a magnitude such that

$$(E : B) :: (C : D);$$

therefore (E. x. 5.) $A > E$;

$$\therefore A+B > E+B;$$

therefore (E. viii. 5.) $(A+B : B) > (E+B : B)$:

But (E. xviii. 5. and *hyp*.)

$$(E+B : B) :: (C+D : D);$$

therefore (E. xi. 5.) $(A+B : B) > (C+D : D)$.

PROP. XI.

12. THEOREM. *If the first term of a ratio be less than the second, the ratio shall be increased by adding the same quantity to both terms; but if the first term be greater than the second, the ratio shall be diminished by adding the same quantity to both.*

Let $A < B$, and let C be any other magnitude:

Then is $(A+C : B+C) > (A : B)$.

For, (E. xviii. 5. and *hyp*.), $(C : A) > (C : B)$;

therefore (Supp. x. 5.), $(A+C : A) > (B+C : B)$;

therefore (Supp. vii. 5.) $(A+C : B+C) > (A : B)$.

And, if A be greater than B, it may, in the same manner, be shewn that

$$(A+C : B+C) > (A : B).$$

13. **Cor.** If $a < A$ and $b < B$, a fourth proportional to $A+B$, A and B, shall be greater than a fourth proportional to $a+b$, a and b.

For, let $A+b : A :: b : D$, and $a+b : a :: b : d$. And, since (12) $(a+b : a) > (A+b : A)$ $D > d$. In like manner, if $A+B : B :: A : E$, it may be shewn that $E > D$: Much more, then, is $E > d$.

Prop. XII.

14. Theorem. *If the first of four magnitudes, of the same kind, have a greater ratio to the second than the third has to the fourth, the first, together with the third, shall have to the second, together with the fourth, a greater ratio than the third has to the fourth, and a less ratio than the first has to the second.*

If $(A : B)$ be greater than $(C : D)$, then is

$$(A+C : B+D) > (C : D);$$

$$\text{and } (A+C : B+D) < (A : B).$$

For, (Supp. vii. 5. and *hyp.*) $(A : C) > (A : D)$; therefore (Supp. x. 5.), $(A+C : C) > (B+D : D)$; therefore (Supp. vii. 5.), $(A+C : B+D) > (C : D)$:

Again, since (*hyp.* and Supp. vi. 5.), $(B : A) < (D : C)$, or $(D : C) > (B : A)$, it may be shewn, in the same manner, that

$$(A+C : B+D) < A : B.$$

PROP. XIII.

15. THEOREM. *If the first, together with the second, have to the second, a greater ratio than the third, together with the fourth, has to the fourth, then shall the first, together with the second, have to the first, a less ratio than the third, together with the fourth, has to the third.*

If $(A+B : B)$ be greater than $(C+D : D)$, then is $(A+B : A) < (C+D : C)$.

For (Supp. ix. 5. and *hyp.*) $(A : B) > (C : D)$;
therefore (Supp. vi. 5.) $(B : A) < (C : D)$;
therefore (Supp. x. 5.) $(A+B : A) < (C+D : C)$.

PROP. XIV.

16. THEOREM. *If the first, together with the second, have to the third, together with the fourth, a greater ratio than the first has to the third, then shall the second have to the fourth a greater ratio, than the first, together with the second, has to the third, together with the fourth.*

If $(A+B : C+D)$ be greater than $(A : C)$, then is $(B : D) > (A+B : C+D)$.

For (*hyp.* and Supp. vii. 5.)
$(A+B : A) > (C+D : C)$;
therefore (Supp. xiii. 5.) $(A+B : B) < (C+D : C)$;
therefore (Supp. vii. 5.) $(A+B : C+D) < (B : D)$;
Or, $(B : D) > (A+B : C+D)$.

PROP. XV.

17. THEOREM. *If four magnitudes be propor-*

tionals, they are also proportionals by conversion: that is, the first is to its excess above the second, as the third to its excess above the fourth.

Let $A+B:B::C+D:D$; then $A+B:A::C+D:C$.

For (*dividendo*) $A:B::C:D$;
therefore (*invertendo*) $B:A::D:C$;
therefore (*componendo*) $A+B:A::C+D:C$.

PROP. XVI.

18. **THEOREM.** *If the first of four magnitudes of the same kind, have a greater ratio to the second, than the third has to the fourth, and if the aggregate of the first and second be greater than the aggregate of the third and fourth, the first shall be greater than the third.*

Let $(A:B)>(C:D)$, and let $(A+B)>(C+D)$; then $A>C$.

For let E be a magnitude, such that
$A:B::C:E$; then (*hyp.* and E. xiii. 5.)
$(C:E)>(C:D)$; therefore (E. x. 5.) $E<D$;
and since (*hyp.*) $A:B::C:E$,
therefore (*componendo* and *convertendo*)
$A+B:A::C+E:C$; but (*hyp.*) $(A+B)>(C+D)$;
$\therefore$ $(A+B)>(C+E)$; therefore (E. xiv. 5.) $A>C$.

PROP. XVII.

19. **THEOREM.** *If any number of magnitudes be continual proportionals, their differences shall, also, be proportionals.*

Let $A : B :: B : C :: D$, &c., then shall

$A-B : B-C :: B-C : C-D$, and so on.

For (*hyp.* and E. xix. 5.)

$$A : B :: A-B : B-C;$$

and $B : C :: B-C : C-D$;

therefore (*hyp.* and E. xi. 5. *Cor.*)

$$A-B : B-C :: B-C : C-D.$$

20. COR. From the demonstration it is manifest, that, if three magnitudes, A, B, C, are proportionals, the excess of the greatest, A, above the mean B, is greater than the excess of the mean B above the least, C.

For it has been shewn that $A : B :: A-B : B-C$;

And (*hyp.*) $A>B$; therefore (Supp. i. 5.)

$$A-B > B-C.$$

PROP. XVIII.

21. THEOREM. *If there be three magnitudes, and other three, and if the first have a greater ratio to the second, in the former set, than the first has to the second, in the latter; and if, also, the second have to the third, in the former set, a greater ratio than the second has to the third, in the latter; then shall the first have a greater ratio to the third, in the former set, than the first has to the third, in the latter.*

Let A, B, C, be three magnitudes, and D, E, F, three other magnitudes: If $(A : B)$ be greater than $(D : E)$, and $(B : C)$ greater than $(E : F)$, then is $(A : C) > (D : F)$.

For let G be a magnitude such that $(G : C) :: (E : F)$;

therefore (*hyp.* and E. x. 5.) $B > G$;
therefore (E. viii. 5.), $(A : G) > (A : B)$.

Again, let H be a magnitude such that $(H : G) :: (D : E)$;

therefore (*hyp.* and E. xiii. 5.) $(H : G) < (A : B)$;
Much more then is $(H : G) < (A : G)$;
therefore (E. x. 5.), $A > H$;
therefore (E. viii. 5.), $(A : C) > (H : C)$;

But (*hyp.* and E. xxii. 5.), $(H : C) :: (D : F)$;

$\therefore (A : C) > (D : F)$.

PROP. XIX.

21. THEOREM. *If there be three magnitudes, and other three, and if the first have to the second, in the former set, a greater ratio than the second has to the third, in the latter; and if, also, the second have to the third, in the former set, a greater ratio than the first has to the second, in the latter; then shall the first have to the third, in the former set, a greater ratio, than the first has to the third, in the latter.*

Let A, B, C, be three magnitudes, and D, E, F, three other magnitudes: If $(A : B)$ be greater than $(E : F)$, and $(B : C)$ greater than $(D : E)$, then is $(A : C) > (D : F)$.

For let G be a magnitude such that $(G : C) :: (D : E)$;

therefore (*hyp.* and E. x. 5.) $B > G$;
therefore (E. viii. 5.), $(A : G) > (A : B)$.

Again, let H be a magnitude such that $(H : G) :: (E : F)$;

therefore (*hyp.* and E. xiii. 5.), $(H : G) < (A : G)$;
therefore (E. x. 5,), $A > H$;
therefore (E. viii. 5.), $(A : C) > (H : C)$;
But (*hyp.* and E. xxiii. 5.) $(H : C) :: (D : F)$;
$\therefore (A : C) > (D : F)$.

Prop. XX.

22. Theorem. *If three magnitudes be proportionals, the two extremes are, together, greater than the double of the mean.*

Let A, B, C, be three magnitudes which are proportionals: Then $A + C > 2B$.

For (*hyp.* and E. vi. Def. Book v.), $(A : B) :: (B : C)$;

therefore (E. xxv. 5.) $A + C > B + B$,
that is, $A + C > 2B$.

23. Cor. An arithmetic mean proportional, between two given magnitudes, is greater than a geometric mean proportional between the same two magnitudes.

Prop. XXI.

24. Theorem. *If there be two sets of magnitudes, the one geometric, and the other arithmetic, proportionals, and if the two first magnitudes be the same in both, any other magnitude in the former set, shall be greater than the corresponding magnitude in the latter.*

Let the magnitudes A, B, C, D, E, &c. be geometric proportionals; and let the magnitudes A, B, c, d, e, &c. be arithmetic proportionals; then is $C>c$, $D>d$, $E>e$, and so on.

For, first, let A be the least magnitude, in each series;

therefore (Supp. xvii. 5. *Cor.*) $C-B>B-A$:

But, from the property of arithmetic proportion,

$$B-A=c-B;$$
$$\therefore\ C-B>c-B;$$
$$\therefore\ C>c.$$

Again, (Supp. xvii. 5. *Cor.*) $D-C>C-B$, and as hath been shewn, $C-B>c-B$ or $d-c$; $\therefore$ $D-C>d-c$; and $C>c$; $\therefore$ $D>d$. In the same manner it may be shewn that $E>e$, and so on.

Secondly, let A be the greatest magnitude in each series:

Then (Supp. xv. 5. *Cor.*) $A-B>B-C$;

But, from the property of arithmetic proportion,

$$A-B=B-c;$$
$$\therefore\ B-c>B-C;$$
$$\therefore\ C>c.$$

Again, (Supp. xvii. 5. *Cor.*) $B-C>C-D$; and it has been shewn that $C>c$; much more then is $B-c>C-D$:

$$\text{But } B-c=c-d;$$
$$c-d>C-D;$$
$$D>d.$$

And, in the same manner, it may be shewn that, in this case, also, $E>e$, and so on.

25. Cor. The two first magnitudes, in both the sets, being the same, if the second of the geometric proportionals be greater than the second of the arithmetic proportionals, then, much more, will every other magnitude, in the former set, be greater than the corresponding magnitude in the latter.

Prop. XXII.

26. Theorem. *If there be two series of magnitudes, the one arithmetically proportional, the other geometrically proportional, but each having the same magnitude for its first term, and if the last term of the arithmetic series be not less than the last term of the geometric series, any other term of the former series shall be greater than the corresponding term in the latter.*

Let the magnitudes A, B, C, D, E, &c. Q, be geometric proportionals; and let the magnitudes A, b, c, d, e, &c. q, be arithmetic proportionals; then if q be not less than Q, $b>B$, $c>C$, $d>D$, and so on.

For (Supp. xxi. 5. and *Cor.*) if B be equal to b, or greater than b, $Q>q$; which is contrary to the hypothesis; $\therefore b>B$.

Again, in the two series B, C, D, &c. Q, b, c, d, &c. q, let b, which has been shewn to be greater than B, be supposed to become equal to B, and q to remain of a magnitude not less than Q; then it is manifest, from the nature of arithmetic proportion, that the intermediate terms C, d, &c. must each, also, become less than they are in the given arithmetic series; and yet, as hath been shewn, the second of them will be greater than C; much more, then, is the term c, in that given series, greater than C: And, in the same manner, it may be proved that $d>D$; and so on.

A

SUPPLEMENT

TO THE

ELEMENTS OF EUCLID.

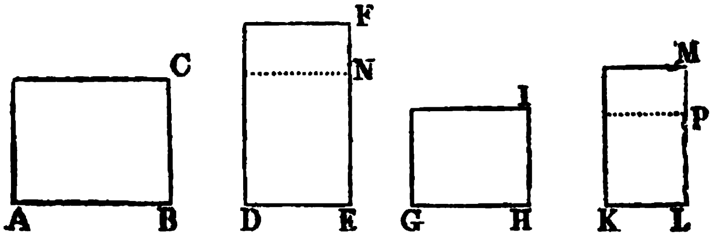
PART I.

BOOK VI.

PROP. I.

1. **THEOREM.** *If the bases of four rectangles be proportionals, and their altitudes be also proportionals, the rectangles themselves shall likewise be proportionals.*

Let the four rectangles *AC, DF, GI, KM,* have



their bases *AB, DE, GH, KL* proportionals, and let their altitudes BC, EF, HI, LM, also, be proportionals: Then $AC : DF :: GI : KM$.

For, in $\overline{EF}$ and $\overline{LM}$, produced if necessary, take $\overline{EN}=\overline{BC}$, and $\overline{LP}=\overline{HI}$, and complete the rectangles DN and KP.

Then since (*hyp.*) $\overline{AB} : \overline{DE} :: \overline{GH} : \overline{KL}$,
therefore (*constr.* E. i. 6. and E. xi. 5.)
$AC : DN :: GI : KP$;
Also, (*hyp.* and *constr.*) $EN : EF :: LP : LM$;
∴ (E. i. 5. and E. xi. 5.) $DN : DF :: KP : KM$;
therefore (E. xxii. 5.) $AC : DF :: GI : LM$.

2. Cor. 1. If four straight lines be proportionals, their squares shall also be proportionals.

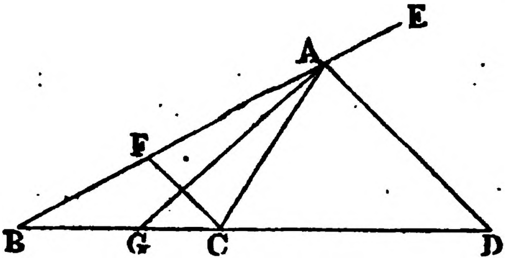
3. Cor. 2. Conversely, if four squares be proportionals, their sides shall likewise be proportionals.

Prop. II.

4. Theorem. *If the outward angle of a triangle, made by producing one of its sides, be divided into two equal angles, by a straight line which also cuts the base produced, the segments between the dividing line and the extremities of the base have the same ratio, which the other sides of the triangle have to one another: And if the segments of the base, produced, have the same ratio which the other sides of the triangle have, the straight line, drawn from the vertex to the point of section, divides the outward angle of the triangle into two equal angles.*

First, let the outward $\angle CAE$, of any $\triangle ABC$, be divided into two equal angles, by $\overline{AD}$, which cuts the base BC, produced, in D: Then $BD : DC :: BA : AC$.

Through C draw (E. xxxi. 1.) CF parallel to AD;



therefore (E. xxix. 1.) the $\angle ACF = \angle CAD$; but (*hyp.*) the $\angle CAD = \angle DAE$; therefore the $\angle ACF = \angle DAE$. Again (*constr.* and E. xxix. 1.) the $\angle DAE = \angle CFA$; and it has been shewn that the $\angle ACF = \angle DAE$; therefore the $\angle ACF = \angle CFA$, and (E. vi. 1.) $AF = AC$. Also (*constr.* and E. ii. 6.) $BD : DC :: BA : AF$; *i. e.* $BD : DC :: BA : AC$, because $AF = AC$.

Secondly, let $BD : DC :: BA : AC$, and let AD be drawn; then the $\angle CAD = \angle DAE$.

The same construction having been made, since
(*hyp.*) $BD : DC :: BA : AC$,
and (*constr.* and E. ii. 6.) $BD : DC :: BA : AF$,
therefore (E. xi. 5.) $BA : AC :: BA : AF$;
therefore (E. ix. 5.) $AC = AF$.

Wherefore (E. v. 1.) the $\angle AFC = \angle ACF$; but (*constr.* and E. xxix. 1.) the $\angle EAD = \angle AFC$, and the $\angle CAD = \angle ACF$; therefore the $\angle EAD = \angle CAD$.

5. Cor. 1. Hence a given finite straight line may be cut in harmonic proportion.

For let BD be the given finite straight line: Take any point *A*, out of *BD*, and through *A* draw $\overline{BAE}$; join *A*, *D*; at the point *A*, in *DA*, make (E. xxiii. 1.) the $\angle DAC = \angle DAE$, and bisect (E. ix. 1.) the $\angle BAC$ by *AG*: Then is *BD* cut harmonically in the points *G* and *C*.

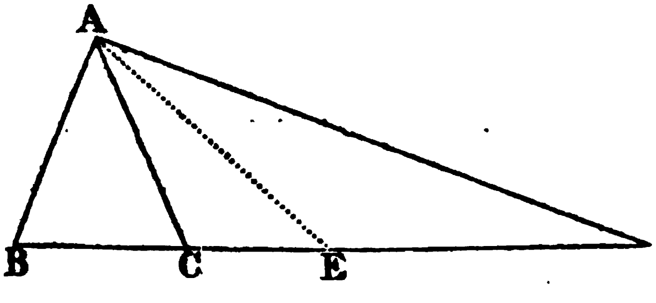
For (*constr.* and E. iii. 6.) $BG : GC :: BA : AC$;
And (*constr.* and Supp. ii. 6.)
$BA : AC :: BD : DC$;
therefore (E. xi. 5.) $BG : GC :: BD : DC$;
therefore (E. xvi. 5.) $BG : BD :: GC : DC$;
that is, $BG : BD :: BC - BG : BD - BC$;
which is the property of harmonic proportion.

6. Cor. 2. If any straight line be drawn between *BE* and *BD*, it may, in the same manner, be shewn to be cut harmonically by the straight lines *AG*, *AC*, and *AD*.

Prop. III.

7. Theorem. *Either of the equal sides of an isosceles triangle, is a mean proportional between the base, and the half of the segment of the base, produced if necessary, which is cut off by a straight line drawn from the vertex at right angles to the equal side.*

Let *ABC* be an isosceles triangle, having the side



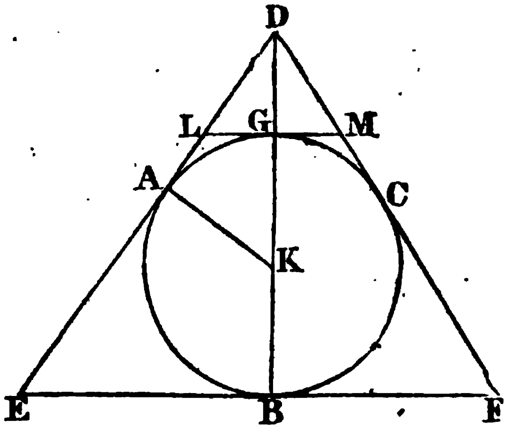
$AB = AC$, and let $\overline{AD}$, drawn perpendicular to AB, meet BC, produced, if it be necessary, in D; also, let BD be bisected in E: Then $BC : AB :: AB : BE$.

For draw $\overline{AE}$; and (Supp. xxix. 1.) $EA = EB$; therefore (E. v. 1.) the $\angle EAB = \angle ABE = \angle ACB$; therefore (E. xxxii. 1.) the $\angle AEB = \angle BAC$; therefore (E. iv. 6.) $CB : BA :: BA : BE$.

PROP. IV.

8. THEOREM. *The diameter of a circle is a mean proportional between the sides of an equilateral triangle and hexagon described about the circle.*

Let DEF be an equilateral triangle, described about the circle ABC, of which the centre is K; let the sides of the $\triangle DEF$ touch the circle in the



points A, B, C; let D, B be joined, cutting the circumference in G, and let $\overline{LM}$ be drawn touching the circle in G; so that (Supp. i. 4. *Cor.* 2.) LM is the side of a regular hexagon described about the circle

ABC, and *GB* passes through the centre *K*; Then, $\overline{DE} : \overline{GB} :: GB : LM$.

For join *A*, *K*; therefore (E. xviii. 1.) the angles *DAK*, *DGL*, are right angles, and the ∠ *ADK* is common to the two △s *DAK*, *DGL*, which (Supp. xxvi. 1.) are, therefore, equiangular;

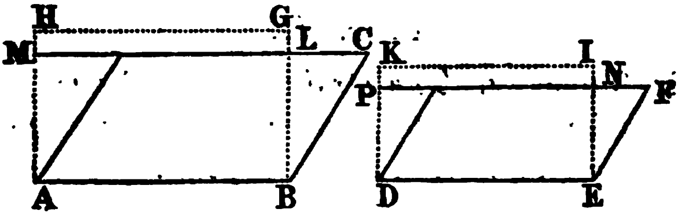
therefore (E. iv. 6.) $\overline{DA} : \overline{AK} :: \overline{DG} : \overline{GL}$.
But (Supp. i. 4. and *Cor.* 1, 2.) *DE* is double of *DA*; the diameter *GB* is double of *AK*, or of *DG*, which (Supp. i. 4. *Cor.* 1.) is equal to *AK*; and *LM* is double of $\overline{LG}$;

therefore (E. xv. 5.) $DE : GB :: GB : LM$.

PROP. V.

9. THEOREM. *Equiangular parallelograms have to one another the same ratio as the rectangles contained by the sides about equal angles in each.*

Let *AC*, *DF*, be two equiangular parallelograms,



having the ∠ *ABC* = ∠ *DEF*:

Then $AC : DF :: \overline{AB} \times \overline{BC} : \overline{DE} \times \overline{EF}$.

For draw (E. xi. 1.) $\overline{BG}$ and $\overline{EI}$ perpendicular to *AB* and *DE* respectively; make *BG* = *BC*, and *EI* = *EF*; and complete the rectangles *ABGH* and *DEIK*; and produce the sides of the given parallelograms, that are opposite to *AB* and *DE*, to meet $\overline{AH}$ and $\overline{DK}$, in *M* and *P*, respectively.

And, since (*hyp.*) the $\angle ABC = \angle DEF$, and (*constr.*) the $\angle ABL = \angle DEN$; therefore the $\angle LBC = \angle NEF$; also (*hyp.*) the $\angle LCB = \angle NFE$; therefore (Supp. xxvi. 1.) the two $\triangle$s LBC, NEF, are equiangular; therefore (E. iv. 1.)

$$BL : BC \text{ or } BG :: EN : EF \text{ or } EI:$$

But (E. i. 6.)

$$BL : BG :: AL : AG;$$

Also, $EN : EI :: DN : DI;$

therefore (E. xi. 1.)

$$AL : AG :: DN : DI;$$

therefore (E. xvi. 5.)

$$AL : DN :: AG : DI:$$

But (E. xxxv. 1.) $\square AL = \square AC$; and $\square DN = \square DF$;

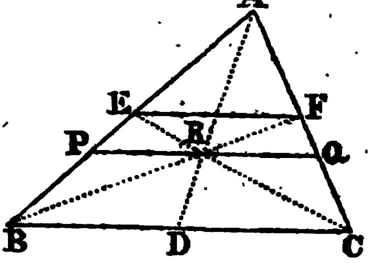
$\therefore AC : DF :: AG \text{ or } AB \times BC : DI \text{ or } DE \times EF.$

10. **Cor.** Triangles, having equal vertical angles, are to one another as the rectangles contained by the sides about those equal angles.

Prop. VI.

11. **Theorem.** *The straight lines, drawn from the bisections of the three sides of a triangle to the opposite angles, meet in the same point.*

Let the sides AB, AC, of the $\triangle ABC$, be bisected



(E. x. 1.) in E and F; and let BF and CE cut one another in the point R: The straight line which is drawn from A, to the bisection of $\overline{BC}$, shall also pass through R.

For join A, R, and produce AR to meet BC in D; join, also, E, F; and through R draw (E. xxxi. 1.) $\overline{PRQ}$ parallel to BC.

And, since (*constr.* and E. ii. 6.) $\overline{EF}$ is parallel to BC; therefore (E. xxix. 1.) the two △s BFE, BRP, are equiangular; as are, also, the △s CEF, CRQ;

therefore (E. ii. 6.) $BF : BR :: CE : CR$;
Also (E. iv. 6.) $BF : BR :: EF : PR$;
And $CE : CR :: EF : RQ$;
therefore (E. xi. 5.) $EF : PR :: EF : RQ$;
therefore (E. ix. 5.) $PR = KQ$;
therefore (Supp. lxi. 1.) $BD = DC$;

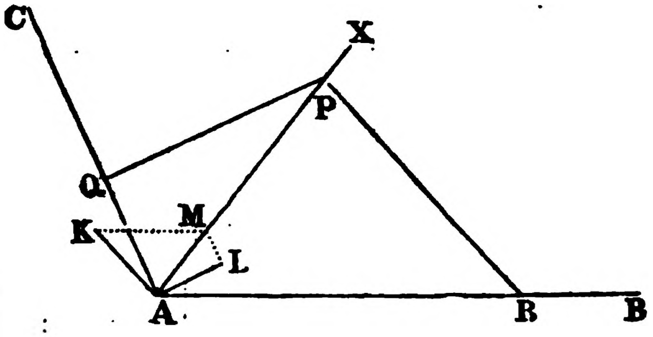
therefore, D is the bisection of BC; and there cannot be two straight lines joining the same two points A and D, which do not coincide; therefore the straight line, drawn from A to the bisection of BC, passes through the point R.

PROP. VII.

12. PROBLEM. *To find, within a given rectilineal angle, first, the* locus *of all the points, from each of which, if two straight lines be drawn, to the lines containing the given angle, so as always to be parallel to two straight lines given in position, they shall be to one another in a given ratio: And secondly, to find the* locus *of all the points, from each of which if two straight lines be drawn in like manner, they shall cut off from two given parts of the straight lines contain-*

ing the given angle, segments that shall be to one another in a given ratio.

Let *CAB* be the given angle; let $\overline{AK}$, $\overline{AL}$, be the



two straight lines given in position; and let *AL* be to *AK* in the given ratio: It is required, first, to find, within the ∠ *CAB*, the *locus* of all the points, from which, if straight lines be drawn to $\overline{AC}$ and $\overline{AB}$, parallel to *AL* and *AK*, respectively, they shall be to one another as *AL* to *AK*.

Through *K* and *L* draw (E. xxxi. 1.) $\overline{KM}$, and $\overline{LM}$, parallel to *AB* and *AC*, respectively, and meeting in *M*; draw $\overline{AM}$, and produce it, indefinitely, toward *X*; $\overline{AX}$ is the *locus* which was to be found.

For take any point *P* in *AX*, and from *P* draw $\overline{PQ}$ parallel to *AL*, and *PR* parallel to *AK*: And, since (*constr.* and E. xxix. 1.) the △s *APR*, *KAM* are equiangular, as are, likewise, the △s *APQ*, *MAL*;

therefore (E. iv. 6.)

$$PR : AK :: AP : AM :: PQ : AL;$$

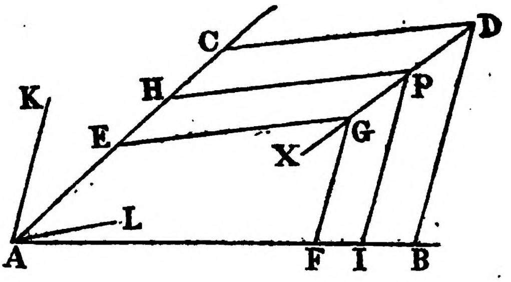
therefore (E. xi. 5.)

$$PR : AK :: PQ : AL;$$

therefore (E. xvi. 5.)

$$PR : PQ :: AK : AL.$$

Secondly, let *B* and *C* be two given points in *AB* and *AC*: It is required to find the *locus* of all the



points, from which if straight lines be drawn parallel to *AK* and *AL*, they shall cut off from *CA* and *BA* two segments, which are always to one another in the same ratio as the given finite straight lines *AK* and *AL*.

From *CA* cut off *CE*=*AK*, and from *BA* cut off *BF*=*AL*; from *C* and *B*, draw (E. xxxi. 1.) *CD* parallel to *AL*, and *BD* parallel to *AK*, and let *CD* and *BD* meet in *D*; likewise from *E* and *F* draw *EG* parallel to *AL* or *CD*, and *FG* parallel to *AK* or *BD*, and let *EG* and *FG* meet in *G*: Through *D* and *G* draw the straight line *DGX*: Then is *DGX* the *locus* which, in this case, was to be found.

For take any point in it, as *P*, and draw *PH* parallel to *DC*, and *PI* parallel to *DB*: Then it is manifest from the demonstration of E. x. 6. that

$$HC : EC :: PD : GD :: IB : FB;$$

therefore (E. xvi. 5.) *HC* : *IB* :: *EC* : *FB*: That is (*constr.*) *HC* is to *IB* in the given ratio: And it is easily shewn, *ex absurdo*, that no point which is out of the *locus* so determined, has the property described in the proposition.

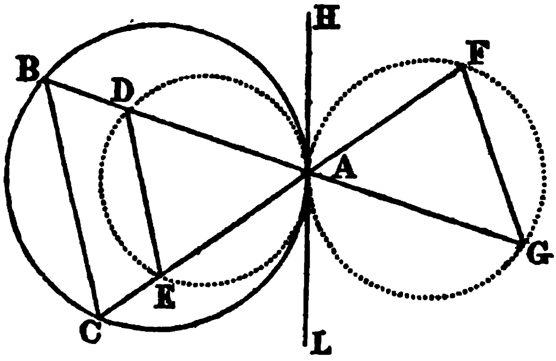
13. Cor. The intersection of the one *locus* with

the other, determines a point, from which if two straight lines be drawn to *AB* and *AC*, in the given directions, they shall be to one another in the same given ratio as the segments are, which they cut off from *CA* and *BA*.

PROP. VIII.

14. THEOREM. *If a circle be touched, in the same point, both externally and internally, by two other circles, and through the point of contact two straight lines be drawn, the parts of them intercepted between the circumference of the given circle, and that of the circle which touches it internally, shall have to one another the same ratio as the parts which are chords of the other circle.*

Let the circle *ABC* be touched in the same point *A*, internally by the circle *DAE*, and externally by the circle *FAG*; and through *A* let there be drawn any two straight lines, *BAG*, *CAF*, each cutting the three circles *ABC*, *DAE*, *FAG*:



Then $BD : CE :: AG : AF$.

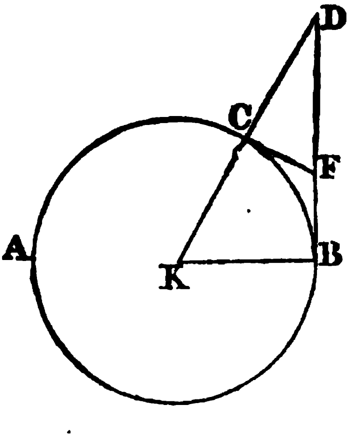
For, draw $\overline{BC}$, $\overline{DE}$, and $\overline{FG}$; and through *A* draw (E. xvii. 1.) *HAL* touching the circle *BAC*, in *A*, and therefore touching the two circles DAE, FAG: And since (E. xv. 1.) the $\angle\, DAH = \angle\, LAG$, and that (E. xxxii. 3.) the $\angle\, DAH = \angle\, DEA$, and the $\angle\, LAG = \angle\, AFG$; therefore the $\angle\, DEF = \angle\, EFG$, and therefore (E. xxvii. 1.) FG is parallel to DE: Also, since (E. xxxii. 3.) the $\angle\, DAH$ or BAH, is equal to each of the $\angle$s DEA, BCA, they are equal to one another, and therefore (E. xxviii. 1.) BC is parallel to DE; therefore (E. ii. 6.)

$$BD : CE :: AG : AF.$$

Prop. IX.

15. Problem. *From the centre of a given circle, to draw a straight line to meet a given tangent to the circle, so that the segment of the line between the circle and the tangent shall be any required part of the tangent.*

Let ABC be a circle, of which K is the centre,



and let $\overline{BD}$ touch the circle in B: It is required to draw a straight line from K to BD, so that the segment of it, between the circle and BD shall be any required part of the tangent BD.

Draw $\overline{KB}$; divide (Supp. xlix. 1.) KB into a number of equal parts, equal to the number of times which the tangent of BD is to contain the segment of the straight line to be drawn from K to BD; and from BD cut off BF equal to one of them; from F draw (E. xvii. 3.) $\overline{FC}$ touching the circle ABC in C; through C draw $\overline{KCD}$: Then shall $\overline{CD}$ be the required part of BD.

For (*constr.* and Supp. xxvi. 1.) the two $\triangle$s KBD, DCF, are equiangular; also (*constr.* and Supp. xix. 3. *Cor.* 1.) $\overline{FB} = FC$;

therefore (E. iv. 1.) $KB : BD :: CF$ or $BF : CD$;
therefore (E. xvi. 5.) $KB : BF :: BD : CD$;
therefore (*constr.* and Supp. iv. 5.) CD is the required part of BD.

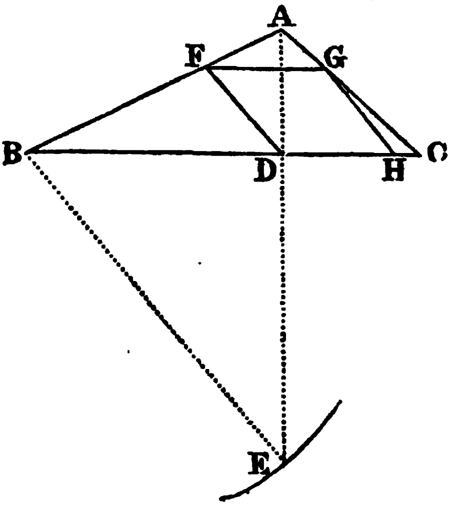
Prop. X.

16. **Problem.** *From a given triangle to cut off a rhombus; the base of the rhombus being part of the base of the triangle, and having its extremity in a given point of that base.*

Let ABC be the triangle, and D the point in its base BC: It is required to cut off from the $\triangle$ ABC a rhombus, having its base in $\overline{BC}$, and terminated by the given point D.

Draw $\overline{AD}$, and produce it; from the centre B, at the distance BC, describe a circle, cutting AD produced in E, and join B, E; $\therefore \overline{BE} = BC$; through D draw (E. xxxi. 1.) $\overline{DF}$ parallel to $\overline{EB}$; also through

F draw *FG* parallel to *BC*, and through *G* draw *GH*



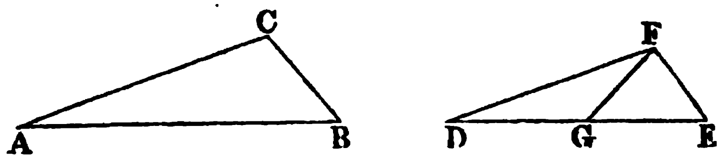
parallel to *FD* or *BE*; therefore the figure *FDHG* is a parallelogram: And since (*constr.* and E. xxviii. 1.) the △s *BAC*, *FAG*, are equiangular, as are, also, the △s *ABE*, *AFD*,

therefore (E. iv. 6.) $AB : BC$ or $BE :: AF : FG$:
And $AB : BE :: AF : FD$;
therefore (E. xi. 5.) $AF : FG :: AF : FD$;
therefore (E. ix. 5.) $FG = FD$:
But (E. xxxiv. 1.) $FG = DH$, and $FD = GH$;

therefore the figure *FDHG*, having its base *DH* in *BC*, and terminated by the point *D*, is a rhombus.

PROP. XI.

17. THEOREM. *If two triangles have one angle of the one, equal to one angle of the other, and also another angle of the one, together with another angle of the other, equal to two right angles, the sides about the two remaining angles shall be proportionals.*



Let the two △s *ABC, DEF,* have the $\angle BAC = \angle EDF$, and another angle, as *ACB,* of the one triangle, together with another angle, as *DEF,* of the other, equal to two right angles:

Then $AB : BC :: DF : FE$.

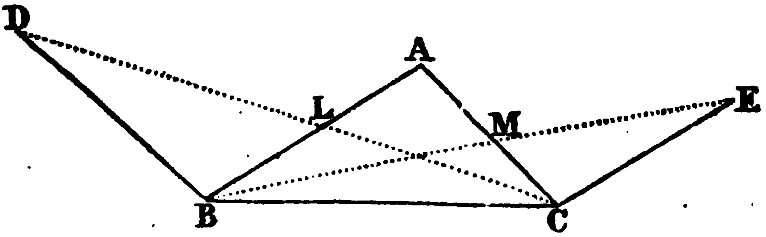
From *F* draw (Supp. xxv. 1.) $\overline{FG}$ making with $\overline{DE}$ an $\angle FGE = \angle FEG$; therefore (E. vi. 1.) $\overline{FG} = \overline{FE}$: And since (*hyp.*) the $\angle ACB + \angle DEF$ = two right angles, and that (E. xiii. 1.) the $\angle DGF + \angle FGE$ = two right angles; therefore the $\angle ACB + \angle DEF = \angle DGF + \angle FGE$; but (*constr.*) the $\angle FGE = \angle FEG$; therefore the $\angle ACB = \angle DGF$; and (*hyp.*) the $\angle BAC = \angle GDF$; therefore (Supp. xxvi. 1.) the two △s *ACB, DGF,* are equiangular; therefore (E. iv. 6.)

$AB : BC :: DF : FG$ or FE.

Prop. XII.

18. Theorem. *If, from the extremities of the base of a given triangle, there be drawn two straight lines, both on the same side of the base, and each equal to the adjacent side, and making with that side an angle equal to the vertical angle of the triangle, then the straight lines which join the extremities of the lines so drawn, and the further extremities of the base, shall cut off, from the sides, equal segments towards the vertex; and each of those segments shall be a mean proportional between the other segments, that are towards the base.*

From the extremities B and C of the base BC, of



the $\triangle\ ABC$, let $\overline{BD}$ be drawn (E. xxxi. 1.) parallel to $\overline{AC}$, and made equal to $\overline{AB}$; and let $\overline{CE}$ be drawn parallel to AB, and made equal to AC; so that (E. xxix. 1.) each of the $\angle$s ABD, ACE, is equal to the vertical $\angle\ BAC$; also, let $\overline{DC}$ and $\overline{EB}$ be drawn, cutting AB and AC in L and M, respectively: Then $\overline{AL}=\overline{AM}$; and $BL : LA :: AM$ or $LA : MC$.

For (*constr.* and E. xv. 1.) the $\triangle$s DLB, ALC, are equiangular, as are, also, the $\triangle$s EMC, AMB;

therefore (E. iv. 6.) DB or $AB : AC :: BL : LA$;
therefore (E. xviii. 5.) $AB + AC : AC :: AB : AL$:
Again (E. iv. 6.) CE or $AC : AB :: CM : MA$;
therefore (E. xviii. 5.) $AC+AB : AB :: AC : AM$;
therefore (E. xvi. 5.) $AC+AB : AC :: AB : AM$;
therefore (E. ix. 5.) $AL=AM$.

Also, since it has been shewn, that

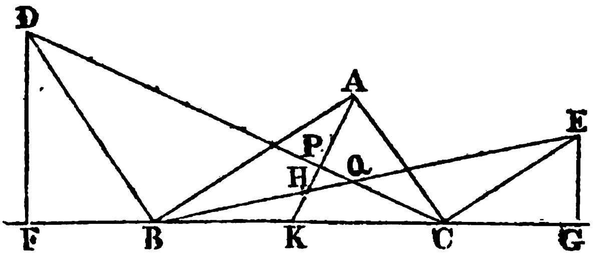
$AB : AC :: BL : LA$,
and $AC : AB :: CM : MA$,
therefore (Supp. ii. 5.) $AB : AC :: AM : CM$;
therefore (E. xi. 5.) $BL : LA :: AM$ or $LA : MC$.

PROP. XIII.

19. **THEOREM.** *If at the extremities of the hypotenuse of a right-angled triangle two straight lines*

be drawn, on the same side of the hypotenuse as the right angle, each equal to, and each perpendicular to, the adjacent side, the two straight lines joining each of their extremities and the further extremity of the hypotenuse, shall cut each other in the same point of the perpendicular drawn to the hypotenuse from the right angle.

From the extremities *A* and *B*, of the hypotenuse



BC of the right-angled $\triangle$ *ABC,* let $\overline{BD}$ and $\overline{CE}$ be drawn perpendicular to *AB* and *AC,* and equal to *AB* and *AC,* respectively, and let $\overline{AK}$ be drawn perpendicular to *BC*: Then if *D, C* and *E, B* be joined, $\overline{DC}$ and $\overline{EB}$ shall cut one another in the same point of *AK.*

For, if it be possible, let *DC* cut *AK* in *P,* and let *EB* cut *AK* in *H*; and from *D* and *E* draw (E. xii. 1.) $\overline{DF}$ and $\overline{EG}$ perpendicular to $\overline{BC}$ produced both ways; therefore (Supp. xxxviii. 1.) $\overline{FB}=\overline{GC}$, and therefore $\overline{FC}=\overline{BG}$: And, since (*constr.*) the $\angle$s *PKC, DFC,* are right angles, and that the angle *PCK* is common to the two $\triangle$s *PCK, DCF*; therefore (Supp. xxvi. 1.) the two $\triangle$s *PCK, DCF* are equiangular; and, in the same manner, the two $\triangle$s *HKB, EGB* may be shewn to be equiangular;

therefore (E. iv. 6.) $CF : FD :: CK : KP$.

But (Supp. xxxviii. 1. *Cor.*)

$$\overline{FD} = \overline{BK}, \text{ and } \overline{CK} = \overline{GE};$$

and it has been shewn that $\overline{CF} = BG$;

$$\text{therefore } BG : BK :: GE : KP;$$

$$\text{But (E. iv. 6.) } BG : BK :: GE : KH;$$

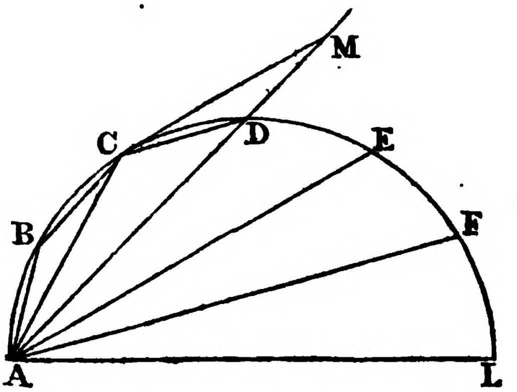
$$\text{therefore (E. ix. 5.) } KH = KP;$$

which is absurd; $\therefore$ DC and EB cannot cut the perpendicular drawn from A to BC, in two different points.

PROP. XIV.

20. THEOREM. *The semi-circumference of a circle having been divided into any number of equal parts, and chords having been drawn, from either extremity of the diameter, to the several points of division, the first chord has to the second, the same ratio which the second has to the aggregate of the first and third; or the same ratio which any other chord has to the aggregate of the two chords that are next to it.*

Let the semi-circumference AEL of a circle, be



divided into any number of equal parts, in the points

B, C, D, E, F, &c.; and let

$$\overline{AB}, \overline{AC}, \overline{AD}, \overline{AE}, \overline{AF}, \text{ \&c.},$$

be drawn: Then

$$AB : AC :: AC : AB+AD :: AD : AC+AE,$$

and so on.

For, from C, as a centre, at the distance CA, describe a circle cutting $\overline{AD}$, produced, in M, and join B, C, and C, D, and C, M; and since (*hyp.*) $\overset{\frown}{AB}=\overset{\frown}{BC}$; therefore (E. xxix. 3.) $AB=\overline{BC}$; also (E. xxvii. 3.) the $\angle BAC=\angle CAD$; therefore (E. v. 1. and Supp. xxvi. 1.) the isosceles $\triangle$s ABC, ACM, are equiangular; therefore (E. iv. 6.)

$$AB : AC :: AC : AM, \text{ or } DM+AD:$$

But (E. xxii. 3.) since $ABCD$ is a quadrilateral figure inscribed in a circle, the $\angle ABC + \angle ADC=$ two right angles; also (E. xiii. 1.) the $\angle ADC + \angle CDM=$ two right angles; therefore the $\angle CDM = \angle ABC$; and the $\angle BAC=\angle CAD$, or (*constr.* and E. v. 1.) $\angle CMD$; and the side CM, of the $\triangle CDM$, is equal to the side CA, of the $\triangle ABC$; therefore (E. xxvi. 1.) $\overline{DM}=\overline{AB}$; and it has been shewn that

$$AB : AC :: AC : DM+AD;$$

therefore $AB : AC :: AC : AB+AD$:

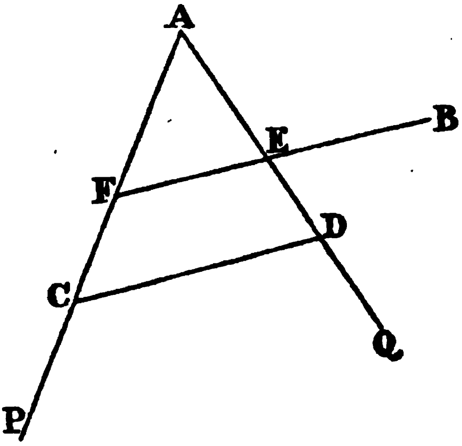
And, by a similar construction, and a similar method of proof, the remaining part of the proposition may be demonstrated.

Prop. XV.

21. Problem. *From a given point, either within or without a given rectilineal angle, to draw a straight line cutting off from the lines which contain*

the angle, segments, towards the summit of the angle, which shall be to one another in a given ratio.

Let PAQ be the given angle, and first let B be a



given point without it: It is required to draw, from B, a straight line which shall cut off from $\overline{AP}$ and $\overline{AQ}$, two segments, towards A, which shall be to one another in a given ratio.

From AP and AQ cut off AC and AD, equal to the two straight lines which exhibit the given ratio, each to each; join D, C; and through B draw (E. xxxi. 1.) $\overline{BEF}$ parallel to $\overline{DC}$: Then, since (*constr.* and E. xxix. 1.) the two △s ADC, AEF are equiangular,

therefore (E. iv. 6.) $AF : AE :: AC : AD$;

therefore (*constr.*) AF is to AE in the given ratio.

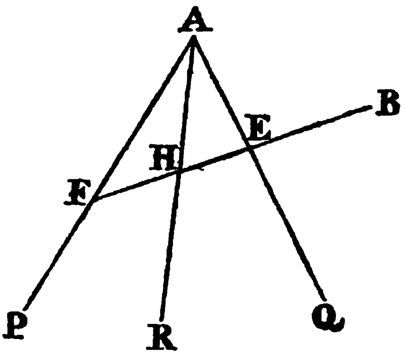
And the problem may be solved in the same manner, when the given point is within the given angle.

Prop. XVI.

22. Problem. *To draw through a given point a straight line cutting the lines which contain a given*

rectilineal angle, so that the segment of it, between those lines, shall be divided by the straight line that bisects the given angle, into two parts, which are to one another in a given ratio.

Let *PAQ* be the given angle; let $\overline{AR}$ be drawn (E. ix. 1.) bisecting it; and let *B* be the given point: It is required to draw, from *B*, a straight line cutting $\overline{AP}$ and $\overline{AQ}$, so that the segment of it, between *AP* and *AQ*, shall be divided by *AR*, into two parts, which are to one another in a given ratio.

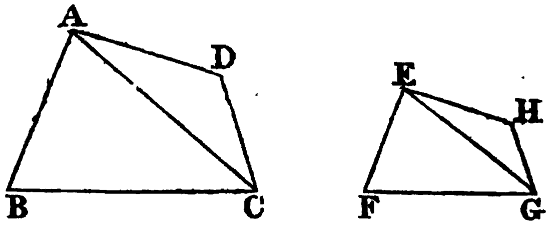


Draw (Supp. xv. 6.) $\overline{BEF}$, so that *AF* shall be to *AE* in the given ratio, and let *BF* cut *AR* in *H*; then, (E. iii. 6.) since $\overline{AH}$ bisects the ∠ *FAE*, *FH* : *HE* :: *AF* : *AE*; that is, *FH* is to *HE* in the given ratio.

PROP. XVII.

23. THEOREM. *If two trapeziums have an angle of the one equal to an angle of the other, and if, also, the sides of the two figures, about each of their angles, be proportionals, the remaining angles of the one shall be equal to the remaining angles of the other.*

Let the two trapeziums *ABCD*, *EFGH*, which



have the sides about each of their angles proportionals, have the ∠ *ABC* equal to the ∠ *EFG*: The two figures shall be equiangular.

For draw $\overline{AC}$ and $\overline{EG}$: Then (*hyp.* and E. vi. 6.) the △s *ABC*, *EFG*, are equiangular, and have their equal angles opposite to the homologous sides.

Therefore (E. iv. 6.) *BA* : *AC* :: *FE* : *EG*;
and (*hyp.*) *DA* : *BA* :: *HE* : *FE*;
therefore (E. xxii. 5.) *DA* : *AC* :: *HE* : *EG*:

And in the same manner it may be shewn, that
DC : *CA* :: *HG* : *GE*:
And (*hyp.*) *AD* : *DC* :: *EH* : *HG*;

therefore (E. v. 6.) the △s *ADC*, *EHG*, are equiangular, and have their equal angles opposite to the homologous sides; and it has been shewn, that the △s *ABC*, *EFG*, are likewise equiangular; therefore the trapeziums *ABCD*, *EFGH* are equiangular.

PROP. XVIII.

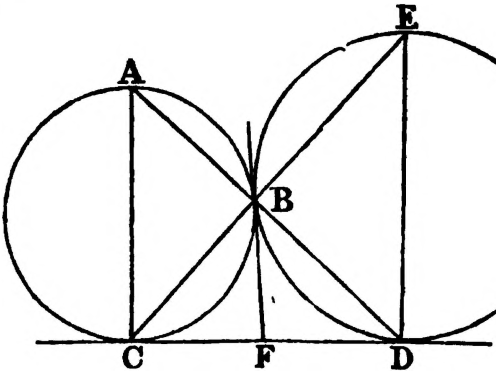
24. THEOREM. *If two straight lines touch a circle at opposite extremities of its diameter, any other tangent of the circle, terminated by them, is so divided in its point of contact, that the radius of the circle is a mean proportional between its segments.*

For (Supp. xx. 3. *Cor.*) the tangent, so terminated, subtends at the centre of the circle a right angle, and (E. xviii. 3.) the straight line drawn from the centre to the point of contact, meets that tangent at right angles, and is, therefore, (E. viii. 6. *Cor.*) a mean proportional between the segments of the tangent.

PROP. XIX.

25. THEOREM. *If two given circles touch each other, and also touch a given straight line, the part of the line between the points of contact, is a mean proportional between the diameters of the circles.*

Let the two circles *ABC, EBD,* which touch one another in *B,* be each of them touched by $\overline{CD}$ in the points *C* and *D*: Then is *CD* a mean proportional between the diameters of the two circles *ABC, EBD.*

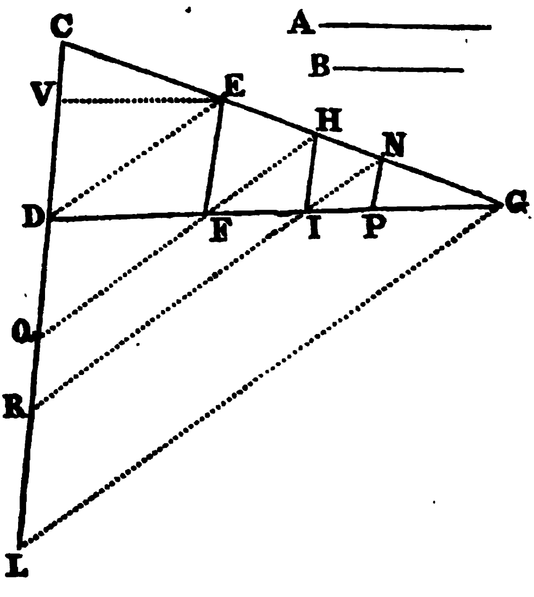


For draw the diameters *CA* and *DE,* which (E. xviii. 3.) are perpendicular to *CD*; also draw (E. xvii. 3.) $\overline{BF}$ touching each of the circles in *B,* and join *A, B,* and *C, B,* and *E, B,* and *D, B*: Then, since (Supp. xix. 3. *Cor.* 2.) $FB = FC$, and $FB = FD$, a circle described from the centre *F,* at the distance *FB*, would pass through *C* and *D*; therefore (E. xxxi. 3.) the ∠ *CBD* is a right angle, as is, also, the ∠ *EBD*; therefore (E. xiv. 1.) $\overline{CB}$ and $\overline{BE}$ are in the same straight line; and, in the same manner, it may be shewn that $\overline{AB}$ and $\overline{BD}$ are in the same straight line; but (E. viii. 6.) the ∠ *CAD* = ∠ *DCB* or *DCE*; therefore (Supp. xxvi. 1.) the two right-angled △s *EDC, DCA* are equiangular; therefore (E. iv. 6.) *ED* : *DC* :: *DC* : *CA*.

PROP. XX.

26. **PROBLEM.** *The two straight lines being given, which are the two first of a series of proportionals, to find the rest; and, if the series decrease, to find a line which shall be greater than the aggregate of any number, whatever, of its terms, but to which the aggregate may approximate indefinitely.*

Let *A*, *B* be the two first of a decreasing series of



proportionals: It is required to find a line which shall be the limit of the aggregate of the proportionals.

Make $\overline{CD}=A$, and $\overline{EF}=B$; and let EF be drawn (E. xxxi. 1.) parallel to CD; join C, E and D, F, and let $\overline{CE}$ and $\overline{DF}$ be produced, so as to meet, in G; join E, D, and through G draw $\overline{GL}$ parallel to $\overline{ED}$, and let it meet CD, produced, in L: Then is $\overline{CL}$ the line which was to be found.

For, through F draw $\overline{HFQ}$ parallel to ED; and through H draw HI parallel to EF or CD:

Then (*constr.* and E. xxxiv. 1.) $DQ=EF$; also, (*constr.* and E. ii. 6.)

$$CD : DQ :: CE : EH :: DF : FI:$$

But since (*constr.* and E. xxvii, xxviii. 1.) the $\triangle$s EFD, HIF are equiangular.

therefore (E. iv. 6.) $DF : FI :: EF : HI$;

therefore (E xi. 5.) $CD : DQ$ or $EF :: EF : HI$:

So that HI is the next of the proportionals to EF; and, by a similar construction, the next of them NP, may be found; and so on:

$$\text{But } CL=CD+DQ+QR+\&c.;$$

and, by the construction, DQ, QR, &c. are equal to the several proportionals: It is manifest, therefore, that CL is their limit.

27. Cor. The first term of a decreasing series of proportionals is a mean between the excess of the first term above the second, and the line which is the limit of all the terms.

For draw $\overline{EV}$ parallel to DG; then since (E. xxix. 1.) the $\triangle$s CVE, CDG are equiangular, as are, also, the $\triangle$s CED, CGL,

therefore (E. iv. 6.) $CV : CD :: CE : CG :: CD : CL$;
therefore (E. xi. 1.) $CV : CD :: CD : CL$.

And, since (E. xxxiv. 1.) $VD = EF$; $\therefore CV = CD - EF$.

PROP. XXI.

28. PROBLEM. *To describe a square which shall have a given ratio to a given rectilineal figure.*

Find (E. xiv. 2.) a square that shall be equal to the rectilineal figure, and from its side, produced if it be necessary, cut off (E. x. 6.) a part, which shall be to the side itself in the given ratio: The rectangle, contained by the side of the square and the part so cut off, will (E. i. 6.) have to the given square the given ratio: If, therefore, lastly, a square be found (E. xiv. 2.) that is equal to the rectangle, it will have to the given square the given ratio.

29. COR. Hence a square may be cut off from a given square, which shall be any required part of it.

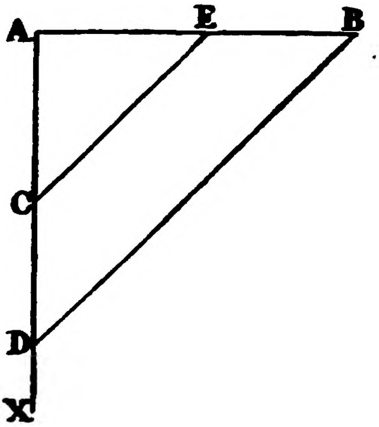
PROP. XXII.

30. PROBLEM.. *To divide a given finite straight line into two parts, the squares of which shall be to one another in a given ratio.*

Let AB be the finite straight line: It is required to divide it into two parts, the squares of which shall be to one another in a given ratio.

Draw (E. xi. 1.) $\overline{AX}$ perpendicular to $\overline{AB}$; find (Supp. xxi. 6.) the sides of two squares, which shall be to one another in the given ratio, and from AX cut off $\overline{AC}$ equal to one of them, and $\overline{CD}$ equal to the other; join D, B; and from C draw (E. xxxi. 1.)

$\overline{CE}$ parallel to $\overline{DB}$: Then is $\overline{AB}$ divided in E, so



that the squares of AE and EB are to one another in the given ratio.

For (*constr.* and E. ii. 6.) $AE : EB :: AC : CD$; therefore (Supp. i. 6. *Cor.* 1.) $\overline{AE}^2 : \overline{EB}^2 :: AC^2 : CD^2$:

But (*constr.*) AC^2 is to CD^2 in the given ratio;

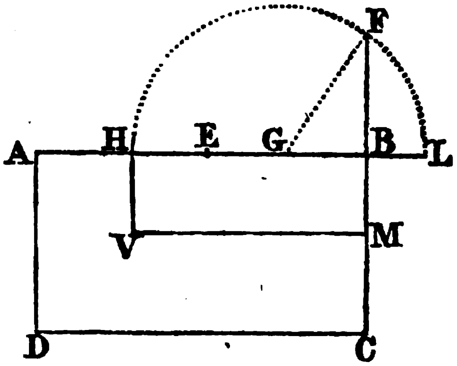
$\therefore$ AE^2 is to EB^2 in the given ratio.

Prop. XXIII.

31. Problem. *To find two points, situated in two adjacent sides of a given oblong, at equal distances from two opposite angles, from which, if two straight lines be drawn parallel to the sides of the figure, they shall cut off from it any part required.*

Let $ABCD$ be the given oblong: It is required to find in two of its adjacent sides, as in AB and BC, two points equidistant from the $\angle$s A and C, from which if straight lines be drawn parallel to $\overline{BC}$ and $\overline{BA}$, they shall cut off a given part of the oblong $ABCD$.

From $\overline{AB}$ cut off $\overline{AE}=BC$; produce $\overline{CB}$; find



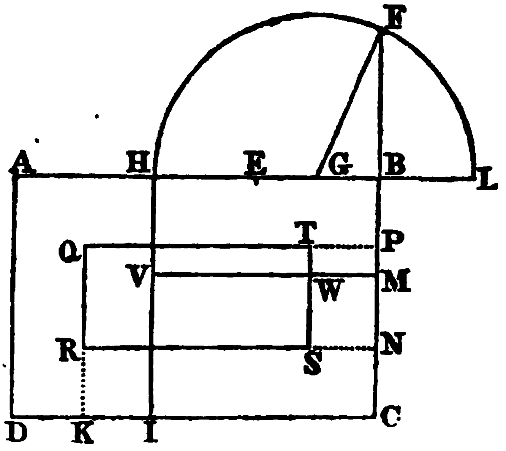
(Supp. xxi. 6.) a square which shall be the same part of the oblong, as that which is to be cut off, and in CB, produced, make $\overline{BF}$ equal to the side of that square; bisect $\overline{EB}$ in G; from the centre G, at the distance $\overline{GF}$, describe the circle HFL, cutting AB produced in L, and AB in H; from CB cut off $\overline{CM} = \overline{AH}$: Then are H and M the points which were to be found.

For, since (*constr.*) $AE = BC$, and $AH = CM$, therefore, $\overline{HE}=\overline{BM}$: Again, since (*constr.*) $\overline{HG}=\overline{GL}$ and $\overline{EG}=\overline{GB}$; ∴ $\overline{HE}=\overline{BL}$; and it has been shewn that $HE=BM$; ∴ $BL=BM$; but (*constr.* and E. xxxv. 3.) $\overline{HB} \times BL=BF^2$; ∴ $HB \times BM=BF^2$; ∴ (*constr.*) $HB \times BM$ is the required part of $AB \times BC$; and the points H and M are equidistant from A and C.

PROP. XXIV.

32. PROBLEM. *Within a given oblong, to describe another oblong which shall be any required part of it, and shall have its four sides all equally distant from the four sides of the given rectangle.*

Let *ABCD* be the given oblong: It is required to describe within it another oblong, which shall be a given part of *ABCD*, and have its sides equally distant from the sides of *ABCD*, each from each.

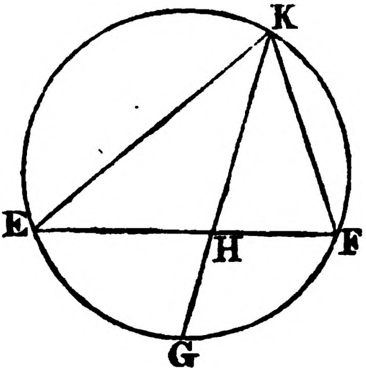


From *ABCD* cut off (Supp. xxiii. 6.) the oblong *HBMV*, the same part of it as that which is to be described, is required to be, and having the extremities *H* and *M*, of its sides *BH* and *BM*, equally distant from *A* and *C*; let $\overline{HV}$, produced, meet $\overline{DC}$ in *I*; bisect (E. x. 1.) $\overline{DI}$ in *K*, and $\overline{CM}$ in *N*; ∴ $\overline{DK}=\overline{CN}$; from *K* draw (E. xxxi. 1.) $\overline{KQ}$ parallel to $\overline{BC}$, and from *N* draw $\overline{NR}$ parallel to $\overline{AB}$; from *BC* and *NR* cut off $\overline{BP}$ and $\overline{NS}$, each equal to *DK* or *CN*; through *P* draw $\overline{PQ}$ parallel to *AC*, and through *S* draw $\overline{ST}$ parallel to *BC*; therefore the figure *QRST* is an oblong: And it is manifest, from the construction, that $\overline{RS}=\overline{HB}$ and $\overline{RQ}=\overline{BM}$, and that, therefore, the gnomon *QRW* is equal to the gnomon *HBW*, for the one may evidently be applied to the other so as to coincide with it; add to these equals the rectangle *VT*, and it is plain that the oblong *QRST* is equal to the oblong *HM*, which was made the required part of the given oblong *ABCD*.

PROP. XXV.

33. PROBLEM. *The base, the vertical angle, and the ratio of the two sides of a triangle being given, to construct it.*

Let *EF* be a straight line: Upon *EF*, as a base, it is required to construct a triangle, having its vertical angle equal to a given angle, and its two remaining sides in a given ratio to one another.



Upon $\overline{EF}$ describe (E. xxxiii. 3.) a segment of a circle *EKF*, capable of containing an angle equal to the given angle, and complete the circle *EKFG*; divide (E. x. 6.) $\overline{EF}$ in *H*, so that $\overline{EH}$ is to $\overline{HF}$ in the given ratio; bisect (E. xxx. 3.) $\overset{\frown}{EGF}$ in *G*; draw $\overline{GH}$, and produce it to meet the circumference in *K*; lastly, join *E*, *K*, and *F*, *K*: Then is *EKF* the triangle which was to be constructed.

For since (*constr.*) $\overset{\frown}{EG} = \overset{\frown}{FG}$; therefore (E. xxvii. 3.) the $\angle EKG = \angle FKG$, so that the $\angle EKF$ is bisected by $\overline{KH}$;

therefore (E. iii. 6.) $KE : KF :: EH : HF$: That is (*constr.*) KE is to KF in the given ratio, and the vertical $\angle EKF$ is equal to the given angle.

PROP. XXVI.

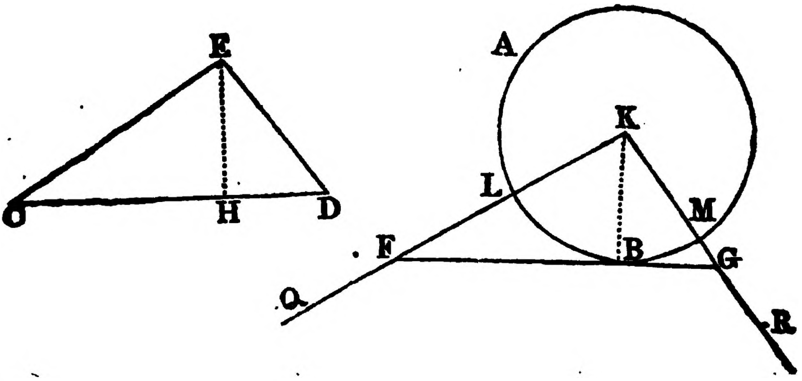
34. **PROBLEM.** *A given finite straight line being divided into any two given parts, to divide it again, so that the rectangle contained by the two former given parts shall have a given ratio to the rectangle contained by the two latter parts.*

Describe (Supp. xxi. 6.) a square which shall be to the rectangle, contained by the given parts of the given line, in the given ratio; and divide (Supp. lxxi. 3.) the given line into two parts, so that the rectangle contained by them shall be equal to the square so described: It is manifest that this rectangle will be to the rectangle, contained by the two given parts, in the given ratio.

PROP. XXVII.

35. **PROBLEM.** *To draw a straight line to touch a given arch of a circle, so that being terminated by the semi-diameters, produced, which bound the arch, it shall be divided by the point of contact, into two parts that are to one another in a given ratio.*

Let LBM be an arch of the circle $ALBM$, terminated by the two semi-diameters KL and KM: It is required to draw a tangent to the circle, so that, being terminated by *KL* and *KM*, produced, it shall be divided, by the point of its contact, into two segments, that are to one another in a given ratio.



Take any straight line *CD*, and divide it (E. x. 6.) in *H* in the given ratio; draw (E. xi. 1.) $\overline{HE}$ perpendicular to $\overline{CD}$, and let *HE* be cut in *E*, by a segment of a circle described (E. xxxiii. 3.) upon *CD*, capable of containing an angle equal to the ∠ *LKM*; and join *C*, *E*, and *D*, *E*; therefore the ∠ *CED* = ∠ *LKM*; lastly, draw (Supp. viii. 3. *Cor.*) $\overline{FBG}$ touching the circle *ALBM*, and making with $\overline{KL}$, produced, an ∠ *KFG* = ∠ *ECD*: Then is the tangent *FG* divided in *B*, so that *FB* is to *BG* in the given ratio.

For join *K*, *B*; therefore (*constr.* and E. xviii. 3.) the angles at *B* are right angles; as are, also, the angles at *H*; and (*constr.*) the ∠ *ECH* = ∠ *KFB*; therefore (Supp. xxvi. 1.) the △s *ECH*, *KFB* are equiangular; and since the ∠ *CEH* = ∠ *FKB*, and that (*constr.*) the whole ∠ *CED* equal whole ∠ *FKG*, therefore the ∠ *HED* = ∠ *BKG*, and (Supp. xxvi. 1.)

the △s *EHD, KBG* are equiangular, as are, likewise, the △s *CED, FKG*;

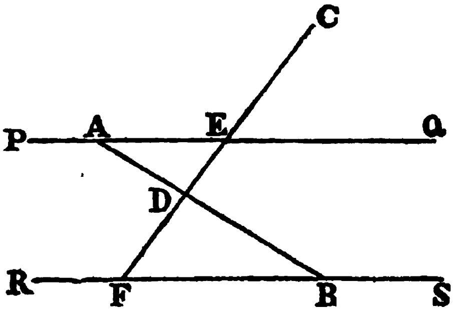
therefore (E. iv. 6.) *CH* : *HE* :: *FB* : *BK*;
and *HE* : *HD* :: *BK* : *BG*;
therefore (E. xxii. 5.) *CH* : *HD* :: *FB* : *BG*:

But (*constr.*) *CH* is to *HD* in the given ratio; therefore *FB* is to *BG* in the given ratio*.

Prop. XXVIII.

36. Problem. *Two points being given, one in each of two parallel straight lines, and a third point being also given, without them, to draw, from that third point, a straight line so to cut the parallels, as that the segments of the parallels, between it and the two first points, shall be to one another in a given ratio.*

Let *PQ* and *RS* be the two given parallel straight lines; *A* and *B* the two given points in them; and



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* Hence, to construct a triangle, of which the vertical angle (*OKR*), the perpendicular (*KB*), and the ratio of the segments of the base (that of *CH* to *HD*) are given.

From the centre *K*, at the distance *K*, *B*, describe the circle *ALM*, and draw the tangent (Prop. 27.) *FG*, divided by the point of contact *B*, so that *FB* : *BG* :: *CH* : *HD*.

C a given point without them: It is required to draw from *C* a straight line cutting $\overline{PQ}$ and $\overline{RS}$, so that the segments of *PQ* and *RS*, between the cutting line and the given points *A* and *B*, shall be to one another in a given ratio.

Join *A*, *B*; and divide (E. x. 6.) *AB* in *D*, so that *AD* is to *DB* in the given ratio; through *D* draw $\overline{CEF}$, cutting *PQ* and *RS*, in *E* and *F*: Then is *CEF* the straight line which was to be drawn.

For, since *PQ* is (*hyp.*) parallel to *RS*, therefore (E. xxix. 1.) the $\angle AEF = \angle EFB$; and the $\angle EAB = \angle ABF$; also (E. xv. 1.) the $\angle ADE = \angle FDB$; so that the △s *ADE*, *BDF* are equiangular;

therefore (E. iv. 6.) $AE : BF :: AD : DB$:

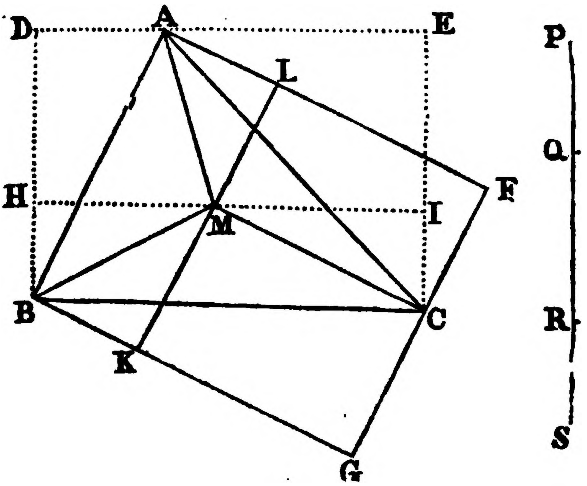
But (*constr.*) *AD* is to *DB* in the given ratio; therefore *AE* is to *BF* in the given ratio.

Prop. XXIX.

37. **Problem.** *To find a point within a given triangle, from which if three straight lines be drawn to the three angles of the triangle, it shall thereby be divided into three parts that are each to each in given ratios.*

Let *ABC* be the given triangle, and let $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, placed in the same straight line, be three given straight lines: It is required to find a point within the △*ABC*, from which if straight lines be drawn to *A*, *B* and *C*, the triangle shall thereby be divided into three parts that are to one another as *PQ*, *QR*, and *RS*.

Through A draw (E. xxxi. 1.) $\overline{DAE}$ parallel to $\overline{BC}$, and from B and C draw (E. xi. 1.) $\overline{BD}$ and $\overline{CE}$ perpendicular to BC; in like manner, describe upon $\overline{AB}$ another rectangle $ABGF$, about the $\triangle ABC$; divide (E. x. 6.) $\overline{DB}$ in H, so that $PS : PQ :: DB : BH$; divide, also, $\overline{BG}$ in K, so that $PS : QR :: BG : BK$; through H draw $\overline{HI}$ parallel to BC, and through K draw $\overline{KL}$ parallel to $\overline{BA}$, and let HI and KL cut one another in M: Then is M the point which was to be found.



For draw $\overline{MA}$, $\overline{MB}$, and $\overline{MC}$: And since (E. xli. 1.) each of the rectangles $DBCE$, $ABGF$, is double of the $\triangle ABC$, they are equal to one another; also (E. xli. 1.) AK is double of the $\triangle AMB$, and HC is double of the $\triangle BMC$.

But (E. i. 6.)

$$HBCI : DBCE :: HB : DB :: PQ : PS;$$

and $ABGF : ABKL :: KB : GB :: PS : QR$;

therefore (E. xxii. 5.) $HBCI : ABKL :: PQ : QR$; therefore (E. xli. 1. and E. xv. 5.)

$$\triangle BMC : \triangle AMB :: PQ : QR.$$

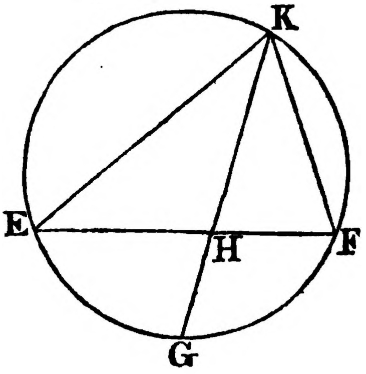
Whence it follows, also, that

$$\triangle AMB : \triangle AMC :: QR : RS.$$

Prop. XXX.

38. **Problem.** *To divide a given circular arch into two parts, so that the chords of those parts shall be to each other in a given ratio.*

Let EKF be the given circular arch: It is required to divide it into two parts, the chords of which shall be to one another in a given ratio.



Join E, F; and describe (E. xxv. 3.) the circle $KEGF$, of which EKF is a given segment; bisect (E. xxx. 3.) $\overset{\frown}{EGF}$ in G; divide (E. x. 6.) $\overline{EF}$ in H, so that $\overline{EH}$ shall be to $\overline{HF}$ in the given ratio; draw $\overline{GH}$, and produce it to meet the circumference in K; lastly join E, K and F, K.

Then, since (*constr.* and E. xxvii. 3.) the $\angle EKF$ is bisected by $\overline{KHG}$, therefore (E. iii. 6.)

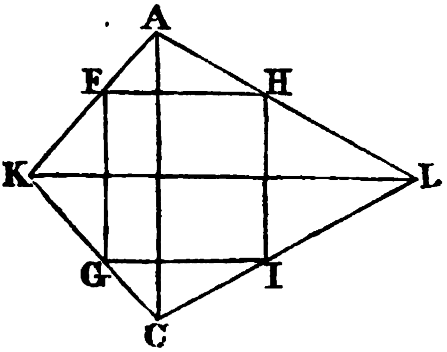
$$\overline{KE} : \overline{KF} :: EH : HF;$$

that is, (*constr.*) $KE : \overline{KF}$ in the given ratio.

PROP. XXXI.

39. PROBLEM. *To inscribe a square in a given trapezium, which has the two sides about any angle equal to one another, and the two sides about the opposite angle also equal to one another.*

Let $AKCL$ be a trapezium having the side



$KA=KC$, and also the side $LA=LC$: It is required to inscribe in $AKCL$ a square.

Draw the diameters of the figure, AC and KL; divide (E. x. 6.) $\overline{AK}$ in F, so that

$$AF : FK :: AC : KL;$$

draw (E. xxxi. 1.) $\overline{FG}$ parallel to $\overline{AC}$, and $\overline{GI}$ and $\overline{FH}$ parallel to $\overline{KL}$; and join H, I: Then is the inscribed figure $FHIG$ a square.

For (Supp. i. 3. *Cor.*) KL bisects AC at right angles; therefore (*constr.* and E. xxxiv. 1.) the angles at F and G are right angles: Again the △s AFH, AKL (E. xxix. 1.) are equiangular, as are, also, the △s KFG, KAC;

therefore (E. iv. 6.) $AK : KL :: AF : FH$:
And (*constr.*) $KL : AC :: KF : AF$;
therefore (E. xxiii. 5.) $AK : AC :: KF : FH$:
But (E. iv. 6.) $AK : AC :: KF : FG$;
therefore (E. ix. 5.) $FG = FH$:

And since (E. ii. 6.) $CG : GK :: AF : FK$, it may, in like manner, be shewn that $GI = \overline{GF}$; and (*constr.*) GI is parallel to FH; therefore (E. xxxiii. 1.) IH is equal and parallel to GF; therefore the figure $FHIG$ is an equilateral parallelogram; and its angles GFH, FGI, have been shewn to be right angles; therefore (E. xxxiv. 1.) all its angles are right angles; therefore (E. xxx. Def. 1.) $FHIG$ is a square.

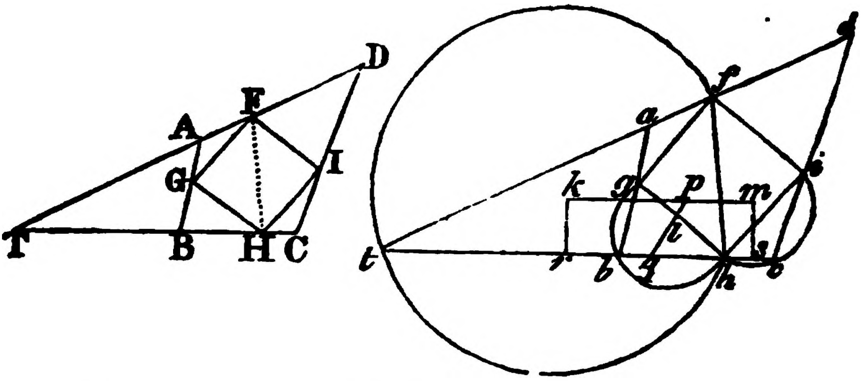
PROP. XXXII.

40. PROBLEM. *To inscribe a square in a given trapezium.*

Let $ABCD$ be the given trapezium: It is required to inscribe in it a square.

Since (E. xxxiv. Def. 1.) $ABCD$ is not a parallelogram, one pair, at least, of its opposite sides must meet if they be far enough produced; let, therefore, DA and CB be produced so as to meet in T: Take any straight line fg and upon it describe (E. xlvi. 1.) the square $fghi$; join f, h; and upon $\overline{hf}$, $\overline{hg}$, and $\overline{hi}$ describe (E. xxxiii. 3.) segments of circles, ich, fth, and gbh, capable of containing angles equal, respectively, to the ∠s T, B, and C, and let k, l, and m, be the several centres of the circles; draw $\overline{km}$, and divide it (E. x. 6.) in p, so that $mp : pk ::$

$CB : BT$; also join p, l; through h draw (E. xii. 1.)



$\overline{chq}$ perpendicular to $\overline{pl}$ produced, and meeting it in q; also let $\overline{cq}$, produced, meet the circumference fth in t, the circumference gbh in b, and the circumference ich in c: Again, divide (E. x. 6.) $\overline{BC}$ in H, so that $BH : HC :: bh : hc$; make (E. xxiii. 1.) at the point H, in $\overline{BH}$, the $\angle BHG = \angle bhg$, the $\angle BHF = \angle bhf$, and the $\angle CHI = \angle chi$; lastly, join F, G and F, I: Then is the inscribed figure $FGHI$ a square.

For draw (E. xii. 1.) $\overline{kr}$ and $\overline{pq}$, perpendicular to $\overline{tc}$: Then, since (*constr.* and E. iii. 3.) $\overline{bh} = 2\overline{qh}$, and $\overline{hc} = 2\overline{hs}$, it is manifest that $\overline{bc} = 2\overline{qs}$; and, in the same manner, it may be shewn that $\overline{tb} = 2\overline{rq}$;

therefore (E. xv. 5.) $tb : bc :: rq : qs$:

But (*constr.* and E. x. 6.)

$rq : qs :: kp : pm :: TB : BC$;

therefore (E. xi. 5.) $tb : bc :: TB : BC$.

Again (*constr.* and Supp. xxvi. 1.) the $\triangle$s gbh, GBH are equiangular, as are, also, the $\triangle$s ich, ICH;

therefore (E. iv. 6.) $hg : hb :: HG : HB$:
And (*constr.*) $hb : hc :: HB : HC$:
Also (E. iv. 6.) $hc : hi :: HC : HI$;
therefore (E. xxii. 5.) $hg : hi :: HG : HI$.

But (*constr.*) $\overline{hg} = \overline{hi}$; $\therefore$ $\overline{HG} = \overline{HI}$; and it is manifest, also, from the construction, that the $\angle GHI = \angle ghi$, of the square $fghi$; therefore the $\angle GHI$ is a right angle.

Again, since (*constr.*) $bh : hc :: BH : HC$, therefore (*comp.* and *div.*) $th : bh :: TH : BH$.

Lastly, (*constr.* and Supp. xxvi. 1.) the two △s tfh, TFH, are equiangular, as are, also, the two △s bhg, BHG;

therefore (E. iv. 6.) $fh : th :: FH : TH$:
And $th : bh :: TH : BH$;
Also (E. iv. 6.) $bh : hg :: BH : HG$;
therefore (E. xxii. 5.) $fh : hg :: FH : HG$.

Wherefore, the two △s fhg, FHG, having their sides about the equal ∠s fhg, FHG, proportionals, are (E. iv. 6.) equiangular; therefore the $\angle FGH$ is a right angle; and (E. iv. 6.) $\overline{FG} = GH$, because (*constr.*) $\overline{fh} = \overline{gh}$: And, as hath been shewn, the ∠s FGH, GHI, are right angles; therefore (E. xxviii. 1.) $\overline{GF}$ is parallel to HI.

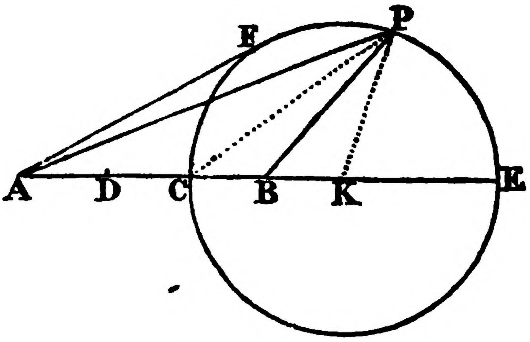
It has been shewn, also, that $HI = HG$; therefore (E. xxxiii. 1. and E. xxxiv. 1.) the figure $FGHI$ is equilateral and rectangular: That is (E. xxx. Def. 1.) it is a square.

PROP. XXXIII.

41. PROBLEM. *To determine the locus of the summits of all the triangles which can be described*

on a given base, so that each of them shall have its two sides in a given ratio.

Let AB be a finite straight line: It is required to determine the *locus* of the summits of all the triangles which can be described upon $\overline{AB}$, as a base, having their two remaining sides, in each, in a given ratio to one another.



Divide (E. x. 6.) AB in C, so that AC shall be to CB in the given ratio; from the greater segment AC, cut off $\overline{CD}=\overline{CB}$; find (E. xi. 6.) a third proportional to $\overline{AD}$ and CB, and in AB, produced, make $\overline{BK}$ equal to it; from the centre K, at the distance KC, describe the circle CPE: The circumference CPE is the *locus* which was to be determined.

For, take any point P, in the circumference CPE, and draw $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, and $\overline{PK}$: Then since;

$$(constr.)\ AD : CB :: CB : BK,$$

therefore (E. xviii. 5.)

$$AD+CB \text{ or } AC : CB :: CK : BK;$$

therefore (E. xvi. 5.) $AC : CK :: CB : BK$;

therefore (E. xviii. 5.) $AK : CK :: CK : BK$;
that is, (E. xv. Def. 1.) $AK : KP :: KP : KB$;
therefore (E. vi. 6.) the two ∠s APK, BPK, are equiangular;
therefore (E. iv. 6.) $PA : PB :: AK : PK$ or CK:
And it has been shewn that

$$AK : CK :: CK : BK :: AC : CB;$$

therefore (E. xi. 5.) $PA : PB :: AC : CB$.
And (*constr.*) AC is to CB in the given ratio; therefore PA is to PB in the given ratio, wherever, in the circumference CPE, the point P is taken*.

Prop. XXXIV.

42. Problem. *The base, the perpendicular distance of the vertex from the base, and the ratio of the two sides of a triangle being given, to construct it.*

Draw (E. xxxi. 1. and E. xi. 1.) a straight line parallel to the given base, and at a perpendicular distance from it equal to the given perpendicular distance; draw, (Supp. xxxiii. 6.) the locus of the summits of all the triangles, which can be described on the given base, having their sides to one another in the given ratio; and it is manifest that the point, in which this locus meets the line drawn parallel to the base, will be the summit of the triangle which was to be described.

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* If the given ratio be a ratio of *equality*, the locus to be determined is, manifestly, the straight line drawn at right angles to AB, through the point which divides AB into two equal parts.

PROP. XXXV.

43. PROBLEM. *The segments into which the perpendicular, drawn from the vertex to the base of a triangle, divides the base, and the ratio of the two remaining sides being given, to construct the triangle.*

The segments being placed in the same straight line, upon their aggregate draw (Supp. xxxiii. 6.) the *locus* of the summits of all the triangles, which can be described on that line, as a base, so as to have their remaining sides in the given ratio: And it is evident that a perpendicular drawn (E. xi. 1.) to this base, from the point, which is common to the two segments, will cut the *locus* in a point, which is the vertex of the triangle that was to be described.

PROP. XXXVI.

44. PROBLEM. *To find a point, from which if three straight lines be drawn to three given points, they shall be each to each in given ratios.*

Upon the straight line joining two of the given points, describe (Supp. xxxiii. 6.) the *locus* of the summits of all the triangles having that line for a base, and having their sides to one another in one of the given ratios; upon the straight line, also, joining the third given point, and either of the other two, describe the *locus* of the summits of all triangles having that line for a base, and having their sides in another of the given ratios: Then it is manifest, that the point, in which the one *locus* cuts the other, is the point which was to be found.

PROP. XXXVII.

45. PROBLEM. *A straight line being divided into three given parts, to find a point without it, at which the three parts shall subtend equal angles.*

Upon the aggregate of the first and second of the given parts, describe (Supp. xxxiii. 6.) the *locus* of the summits of all triangles, having that line for a base, and having their sides to one another, as the first is to the second of the given parts: Again, upon the aggregate of the second and third of the given parts, describe the *locus* of the summits of all triangles having that line for a base, and having their sides to one another as the second of the given parts is to the third: Then it is manifest, from E. iii. 6., that the point, in which the one *locus* cuts the other, is the point which was to be found.

PROP. XXXVIII.

46. PROBLEM. *To find a point in a given line, from which, if two straight lines be drawn to two given points, both on the same side of the given line, they shall be to each other in a given ratio.*

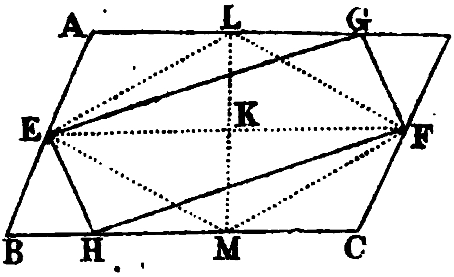
Upon the straight line joining the two given points, describe (Supp. xxxiii. 6.) the *locus* of the summits of all triangles having that line for a base, and having their sides in the given ratio; and it is evident, that the point, in which the locus, so described, cuts the given line, is the point which was to be found.

PROP. XXXIX.

47. PROBLEM. *In a given parallelogram to inscribe a parallelogram that shall have its two adjacent sides in a given ratio to one another, and that shall be the half of the given parallelogram.*

Let $ABCD$ be the given parallelogram: It is required to inscribe it in a parallelogram, which shall be the half of $ABCD$, and which shall have two adjacent sides in a given ratio to one another.

Bisect (E. x. 1.) AB in E, and through E draw (E. xxxi. 1.) $\overline{EF}$ parallel to $\overline{AD}$ or $\overline{BC}$: And, first,



if the given ratio be a ratio of equality, bisect, also, EF in K; through K draw (E. xi. 1.) $\overline{LKM}$ perpendicular to EF; and draw $\overline{EL}$, $\overline{LF}$, $\overline{FM}$, and $\overline{ME}$: Then $ELFM$ is an equilateral parallelogram, and it is the half of the ▭ $ABCD$.

For (E. x. 6.) $\overline{LM}$ is divided, in K, in the same manner as $\overline{AB}$ is divided in E; ∴ $\overline{KL}=\overline{KM}$; therefore (*constr.* and E. iv. 1.) EL, and LF, and FM, and ME, are equal to one another; and therefore (Supp. xviii. 1.) the figure $LEMF$ is a parallelogram: And since (E. iv. 1.) the △*ELF* is the half of the ▭*AEFD*, and the △*EMF* is the half of the ▭*EBCF*, therefore the whole figure *ELFM* is the half of the given ▭*ABCD*.

But, secondly, let the given ratio be not a ratio of equality: In this case, upon *EF* describe (Supp. xxxiii. 6.) the *locus* of all the triangles having *EF* for a base, and having their sides in the given ratio, and let it cut *AD* in *G*; join *E*, *G*, and *F*, *G*; from *CB* cut off *CH*=*AG*, and join *E*, *H* and *F*, *H*: Then is *EGFH* the parallelogram which was to be described.

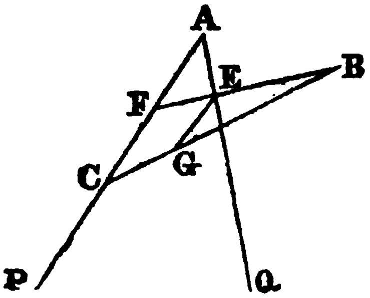
For (*constr.* and Supp. xliii. 1.) *EGFH* is a parallelogram; and it may be shewn to be the half of the given ▭*ABCD*, in the same manner as *ELFM* was shewn to be half of *ABCD*; and (*constr.*) the adjacent sides *EG* and *GF*, are to one another in the given ratio.

Prop. XL.

48. Problem. *From a given point, either within or without a given rectilineal angle, to draw a straight line cutting the two lines which contain the angle, so that the distances of the two intersections from the given point, shall be to one another in a given ratio.*

Let *PAQ* be the given rectilineal angle, and, first, let *B* be a point without it: It is required to draw from *B* a straight line cutting *AP* and *AQ*, so that the distances of its intersections with *AP* and *AQ*, from *B*, shall be to one another in a given ratio.

Through *B* draw *BC* to any point *C* in *AP*;



find (E. xii. 6.) a fourth proportional to the two straight lines, which exhibit the given ratio, and to *BC*; and from *BC* cut off *BG* equal to that fourth proportional; through *G* draw (E. xxxi. 1.) *GE* parallel to *AC*, and meeting *AQ* in *E*; join *B*, *E* and produce it to *F*: Then shall *FB* be to *EB* in the given ratio.

For (*constr.* and E. xxix. 1.) the two △s *BFC*, *BEG*, are equiangular:

therefore (E. iv. 6.) $FB : EB :: CB : GB$:

But (*constr.*) *CB* is to *GB* in the given ratio;
∴ *FB* is to *EB* in the given ratio.

And, by the same method of construction, the problem may be solved, when the given point is within the given angle.

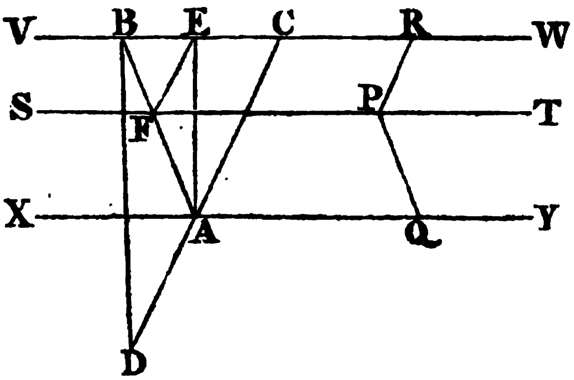
49. COR. It is manifest that the problem admits of the same method of solution if one of the given lines, as *AP*, be a straight line of indefinite length, and if the other *AQ* be a line of any kind, in the same plane with *AP*.

PROP. XLI.

50. PROBLEM. *To find, between two given parallel straight lines, the locus of all the points, from each of*

which if two straight lines be drawn to the two given parallels, so as always to make with them, towards the same parts, given angles, they shall be to one another in a given ratio.

Let $\overline{VW}$ and $\overline{XY}$ be the two given parallels; let



$\overline{AB}$ and $\overline{AC}$, drawn from any point A in XY, be in the two given directions: It is required to find, between VW and XY, a *locus*, from any points of which if two straight lines be drawn to VW and XY, the one parallel to AC and the other parallel to AB, they shall be to one another in a given ratio.

Find (E. xii. 6.) a fourth proportional to the two straight lines, which exhibit the given ratio, and to AB; and from $\overline{CA}$, produced, cut off AD equal to that fourth proportional; join B, D; through A draw (E. xxxi. 1.) $\overline{AE}$ parallel to $\overline{DB}$, and through E draw $\overline{EF}$ parallel to CA, and let it meet AB in F; lastly, through F draw $\overline{SFT}$ parallel to VW or to XY: Then is $\overline{ST}$ the *locus* which was to be found.

For take any point P, in ST, and from P draw $\overline{PQ}$ parallel to AB, and $\overline{PR}$ parallel to AC.

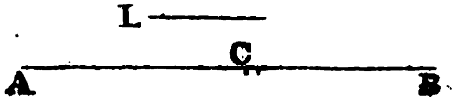
And since (*constr.* and E. xxix. 1.) the △s AEF, ABD are equiangular,

therefore (E. iv. 6.) $FA : FE :: AB : AD$: But (*constr.* and E. xxxiv. 1.) $\overline{FA} = \overline{PQ}$, and $\overline{FE} = PR$; also (*constr.*) AB is to AD in the given ratio; ∴ PQ is to PR in the given ratio.

Prop. XLII.

51. Problem. *To divide a given straight line into two parts, such, that the rectangle contained by the whole line and one of its parts, shall have a given ratio to the square of the other part*[*].

Let AB be the given straight line: It is required



to divide it into two parts, such that the rectangle contained by AB and one of the parts shall have to the square of the other part a given ratio.

Find (E. xii. 6.) a fourth proportional, L, to the two straight lines, which exhibit the given ratio, and to AB; and divide (Supp. lxxxi. 3.) AB into two parts, in C, so that $\overline{AC} \times L = \overline{CB}^2$: And since, (E. i. 6.) $AC \times \overline{AB} : AC \times L$ or $CB^2 :: AB : L$, it is manifest that AB has been divided in C, so that $AC \times AB$ is to CB^2 in the given ratio.

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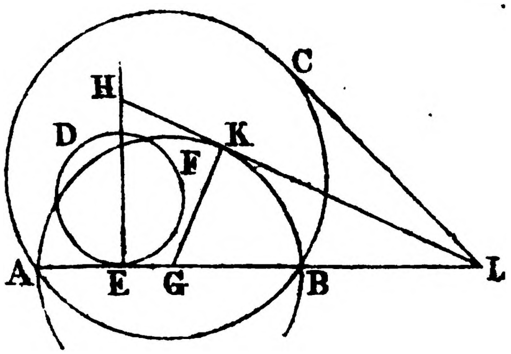
[*] This is the *general* problem, of which E. xxx. 6. is a particular case.

PROP. XLIII.

52. PROBLEM. *One given circle lying within another, to find a point from which, if two tangents be drawn, one to each of the given circles, they shall be to each other in a given ratio.*

Let *ABC, DEF,* be two circles, of which



DEF lies within *ABC*: It is required to find a point from which if tangents be drawn to touch the two circles *ABC, DEF,* they shall be to one another in a given ratio.

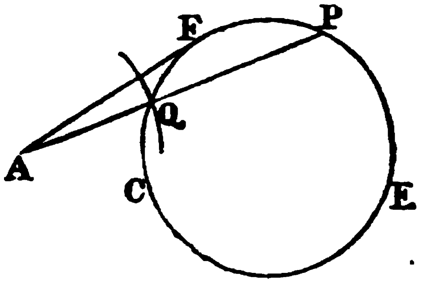
Draw (E. xvii. 3.) *AL* touching the lesser circle *DEF* in any point *E,* and let *AL* meet the circumference of *ABC* in *A* and *B*; bisect (E. x. 1.) *AB* in *G,* and from *E* draw (E. xi. 1.) *EH* perpendicular to *AB*; find (E. xii. 6.) a fourth proportional to the two straight lines, which exhibit the given ratio, and to *AG*; and make *EH* equal to it; from the centre *G,* at the distance *GA* or *GB,* describe the circle *AKB*; from *H* draw (E. xvii. 3.) *HK* touching the circle *AKB* in *K*; and produce *HK* to meet *AB,* produced, in *L*: Then is *L* the point which was to be found.

For, from *L* draw $\overline{LC}$ touching the circle *ABC* in *C*; and join *G*, *K*; therefore (*constr.* and E. xviii. 3.) the $\angle GKL$ is a right angle, as is, also, (*constr.*) the $\angle LEH$; therefore (Supp. xvi. 1.) the $\triangle$s *LKG*, *LEH*, are equiangular;
therefore (E. iv. 6.) $LE : LK :: EH : GK$ or GA:
But, since (E. xxxvi. 3.) $AL \times \overline{LB}$ is equal to $\overline{LK}^2$, and also to LC^2; $\therefore LK = LC$; and (*constr.*) *EH* is to *GA* in the given ratio; therefore the tangent *LE* is to the tangent *LC* in the given ratio.

PROP. XLIV.

53. PROBLEM. *From a given point, to draw a straight line to cut a given circle, so that the distances of the two intersections from the given point, shall be to each other in a given ratio.*

Let *CFE* be the given circle, and *A* the given point without it:. It is required to draw from *A* a straight line cutting *CFE*, so that the distances of its two intersections from *A* shall be to one another in a given ratio.



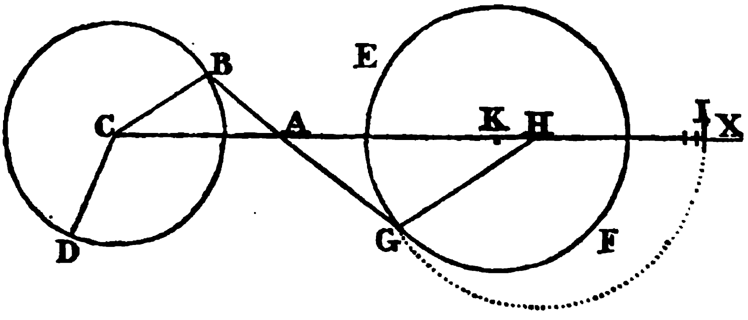
From *A* draw (E. xvii. 3.) $\overline{AF}$ touching the circle *CFE* in *F*; find (Supp. xxi. 6.) a square, which shall be to the square of AF, in the given ratio; from the centre A, at a distance equal to the side of the square thus found, describe a circle cutting the circumference of CFE in Q; and draw $\overline{AQ}$, which is, therefore, equal to the side of that square; produce AQ to meet the circumference of CFE again in P: Then shall AP be to AQ in the given ratio.

For (E. i. 6.) $AP : AQ :: \overline{AP} \times AQ : AQ^2$: But (E. xxxvi. 3.) $AP \times AQ = AF^2$; and (*constr.*) AF^2 is to AQ^2 in the given ratio; $\therefore$ AP is to AQ in the given ratio.

PROP. XLV.

54. PROBLEM. *Two given circles lying wholly without one another, through a given point, which is between the two circles, and which is posited in the straight line joining their centres, to draw a straight line that shall be terminated by the convex circumferences, and divided, by the given point, into two parts, that are to one another in a given ratio.*

Let BD and EF be two circles, and A a point in $\overline{CK}$, which joins the two centres C and K: It is required to draw, through A, a straight line, which being terminated by the convex circum-



ferences of the circles BD and EF, shall be divided by A into two parts, that are to one another in a given ratio.

Produce $\overline{CK}$ indefinitely toward X: Find (E. xii. 6.) a fourth proportional to the two lines, which exhibit the given ratio, and to CA, and from $\overline{AX}$ cut off $\overline{AH}$ equal to it; find, also, a fourth proportional to the same two given lines and any semi-diameter, CD, of the circle BD; and from $\overline{HX}$ cut off $\overline{HI}$ equal to it; from the centre H, at the distance HI, describe a circle, and let it cut the circumference of EF in G; draw $\overline{HG}$, which therefore is equal to HI; draw (E. xxxi. 1.) $\overline{CB}$ parallel to HG; and join B, A, and G, A: Then shall BA and AG be in the same straight line;

for (*constr.* and E. xi. 5.) $CA : AH :: CB : GH$;

therefore (E. xvi. 5.) $CA : CB :: AH : GH$;

and (*constr.* and E. xxix. 1.) the $\angle BCA = \angle AHG$, and two remaining $\angle$s BAC, HAG, of the $\triangle$s CBA, AGH, are of the same species, each of them being, necessarily, less than a right angle; therefore (E. vii. 6.) the $\angle BAC = \angle GAH$; therefore BA and AG are in the same straight line; otherwise (E. xv. 1.) the greater of two angles would be equal to the less:

And since (E. vii. 6.) the two $\triangle$s CBA, AGH, are equiangular,

therefore (E. iv. 6.) $BA : AG :: CA : AH$;

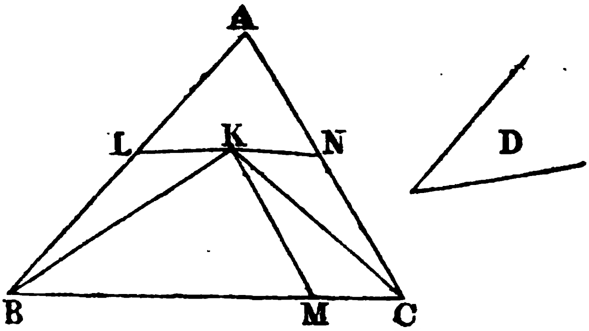
that is, (*constr.*) BA is to AG in the given ratio.

PROP. XLVI.

55. PROBLEM. *To find a point, from which if three straight lines be drawn to meet as many given*

straight lines, which cut one another, so as to make, each with the line on which it falls, an angle equal to a given angle, the lines so drawn shall be, each to each, in given ratios.

Let *AB*, *BC*, and *CA*, be the three given straight



lines, and *D* the given angle: It is required to find a point from which if three straight lines be drawn to *AB*, *BC*, and *CA*, each making with each an angle equal to the ∠ *D*, they shall be to one another in given ratios.

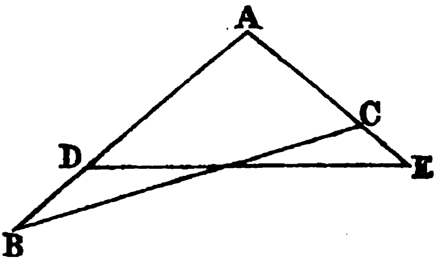
Straight lines being supposed to be drawn at the point *B* making, with *AB* and *BC*, angles each equal to the ∠ *D*, draw (Supp. vii. 6.) the locus *BK* of all the points from which if parallels be drawn to them meeting *AB* and *BC*, these parallels shall be to one another in the first of the given ratios; then (E. xxix. 1.) shall the parallels so drawn make with *AB* and *BC*, angles each equal to the given ∠ *D*.

In like manner draw the locus *CK* of all the points from which if straight lines be drawn to *AC* and *CB*, making with them angles equal each to the ∠ *D*, they shall be to one another in the second of the given ratios: And let *BK* and *CK* meet in *K*. It is manifest that *K* is the point which was to be found.

Prop. XLVII.

56. Problem. *To make an isosceles triangle, which shall be equal to a scalene triangle, and shall also have an equal vertical angle with it.*

Let *ABC* be the scalene triangle: It is required



to make an equal isosceles triangle, which shall have the ∠ *BAC* for its vertical angle.

Find (E. xiii. 6.) a mean proportional between the two unequal sides *AB* and *AC*, of the given △ *ABC*, and from *AB*, the greater side, cut off *AD* equal to the mean proportional so found; also produce *AC* to *E*, so that *AE* = *AD*, and join *D*, *E*: Then is *ADE* the triangle which was to be described.

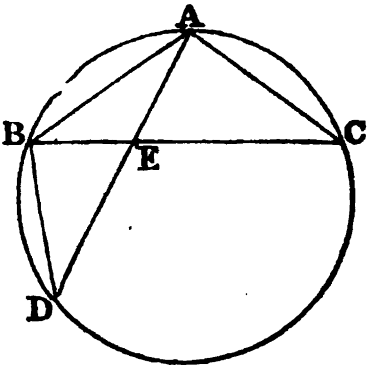
For (*constr.*) *BA* : *AE* :: *AD* : *AC*; therefore (E. xv. 6.) the isosceles △ *ADE* is equal to the △ *ABC*.

Prop. XLVIII.

57. Theorem. *If a straight line, drawn from the vertex of an isosceles triangle cutting the base, be produced to meet the circumference of a circle described about the triangle, the rectangle contained*

by the whole line so produced, and the part of it between the vertex and the base, shall be equal to the square of either of the equal sides of the triangle.

Let $\overline{AD}$ drawn from the vertex, *A*, of the isosceles



$\triangle$ *ABC*, inscribed in the circle *ABDC*, cut the base of the triangle in *E*, and the circumference of the circle in *D*:

$$\text{Then } \overline{DA} \times \overline{AE} = \overline{AB}^2.$$

For join *B*, *D*, and since (*hyp.*) $\overline{AB} = \overline{AC}$, therefore (E. xxviii. 3.) $\overset{\frown}{AB} = \overset{\frown}{AC}$, and therefore (E. xxvii. 3.) the $\angle$ *BDA*, of the $\triangle$ *ABD*, is equal to the $\angle$ *ABE*, of the $\triangle$ *AEB*; and the $\angle$ *BAD* is common to the two triangles; therefore (Supp. xxvi. 1.) they are equiangular;

therefore (E. iv. 6.) $DA : AB :: AB : AE$;

therefore (E. xvii. 6.) $\overline{DA} \times \overline{AE} = \overline{AB}^2$.

PROP. XLIX.

58. THEOREM. *If from a given point, without a circle, two straight lines be drawn to the concave circumference, they shall be reciprocally proportional to the parts of them between the given point and the convex circumference.*

For (E. xxvi. 3. *Cor.*) the rectangle contained by the one of the lines, so drawn, and the part of it without the circle, is equal to the rectangle contained by the other line and the part of it without the circle; therefore (E. xvi. 6.) the two straight lines so drawn are reciprocally proportional to the parts of them, between the point and the convex circumference.

Prop. L.

59. Problem. *To divide a given finite straight line into two parts, such that another given straight line, not greater than the half of the former, shall be a mean proportional between them.*

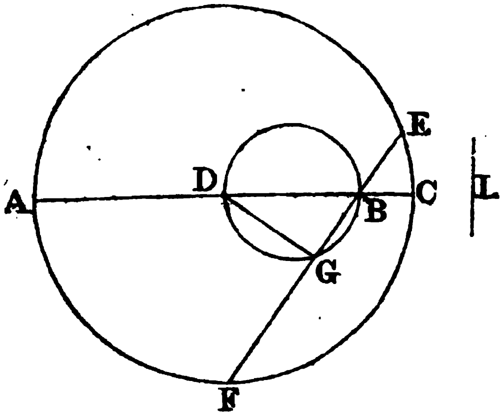
Make (Supp. lxxi. 3.) a rectangle, which shall have its two adjacent sides, together, equal to the given straight line, that is to be divided, and which shall be equal to the square of the other given straight line. Then (E. xvii. 6.) shall this latter be a mean proportional between the parts of the former straight lines, that are equal to the adjacent sides of the rectangle.

Prop. LI.

60. Problem. *Of four straight lines which are continual proportionals, the two extremes being given, and also a line which is equal to the difference of the other two, to find those two lines.*

Let AB and BC, placed in the same straight line, be the two extremes, and L the difference of the two mean terms, of four proportionals: It is required to determine the two mean terms.

Bisect (E. x. 1.) *AC* in *D*, and from the centre *D*,



at the distance *DA* or *DC*, describe the circle *AECF*; likewise, upon *DB*, as a diameter, describe the circle *DGB*; and, since (Supp. iv. 5. *Cor.* and *hyp.*) the double of *DB* is greater than *L*, in the circle *DGB* place (E. i. 4.) *BG* equal to the half of *L*; and produce *GB* both ways to meet the circumference in *E* and *F*: Then are *BE* and *BF* the two mean proportionals, which were to be found.

For join *D*, *G*; and because the ∠ *DGB* is in a semi-circle, it is (E. xxxi. 3.) a right angle; therefore (E. iii. 3.) $GF = GE$; whence it is manifest that the double of *BG*, which was made equal to the half of *L*, is the difference between *BF* and *BE*; also (E. xxxv. 3.)

$$\overline{AB} \times \overline{BC} = \overline{BF} \times \overline{BE};$$

therefore (E. xvi. 6.) $AB : BF :: BE : BC$*.

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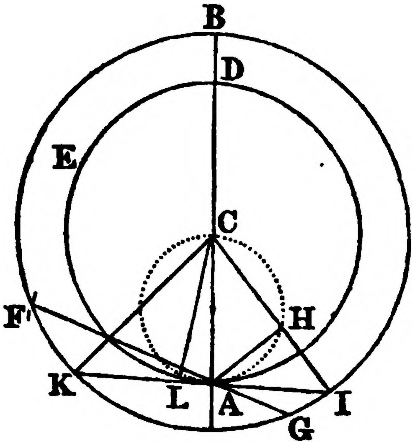
* The method used in this proposition furnishes another, and perhaps a neater, mode of solving the problems contained in Supp. lxxxvi. 3. and its corollary.

Prop. LII.

61. **Problem.** *To make a triangle, which shall have its two sides equal to two given straight lines, each to each, and shall have its base equal to the perpendicular distance of the vertex from the base.*

Let AC and CB be two straight lines: It is



required to describe a triangle, which shall have its base equal to the perpendicular drawn to it from the vertex, and shall have its two remaining sides equal to AC and CB, each to each.

Let AC and CB be placed in the same straight line; and from the centre C, at the distances CA and CB, describe the circles ADE, BFG; from the centre A, at the distance CB, describe a circle cutting the circumference BFG in F; and join A, F; so that $AF = CB$; produce FA to meet the circumference BFG again in G; upon AC as a diameter describe the circle AHC, and in it place $AH = AG$; draw $\overline{CH}$ and produce it to meet the circumference BFG in I; lastly, join I, A, and produce $\overline{IA}$ to meet the circumference *BFG* in *K*: Then is *CAK* the triangle which was to be described.

For, draw (E. xii. 1.) *CL* perpendicular to *AK*; therefore (E. xxxi. 3. and Supp. xxvi. 1.) the △s *CLI*, *AHI*, are equiangular; and since (E. xxxv. 3.)

$$\overline{AI} \times \overline{AK} = \overline{AG} \times \overline{AF},$$

that is, (*constr.*) $AI \times AK = \overline{AH} \times \overline{CK}$,

therefore (E. xvi. 6.) $AI : AH :: CK : AK$:

But (E. iv. 6.) $AI : AH :: CI$ or $CK : CL$;

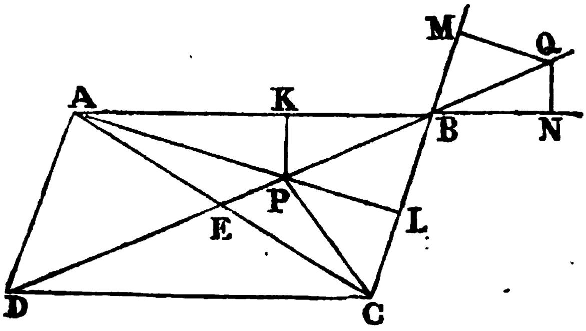
therefore (E. ix. 5.) $AK = CL$.

And the given straight line *AC* is one of the sides of the △ *CAK*; and *CK*, which (E. xv. Def. 1.) is equal to *CB*, is the remaining side.

PROP. LIII.

62. THEOREM. *If from any point in the diameter, or the diameter produced, of a given parallelogram perpendiculars be let fall on the two adjacent sides, produced, if necessary, which meet the diameter, the perpendiculars shall be reciprocally proportional to the sides on which they fall.*

Let *DB* be the diameter of the ▭ *ABCD*; let *P*



be any point in *DB*, and *Q* any point in *DB* produced; and let PK and QN be perpendicular to AB, and PL and QM perpendicular to BC: Then

$$AB : BC :: PL : PK :: QM : QN.$$

For draw $\overline{AC}$, and $\overline{AP}$, and $\overline{PC}$; and let $\overline{AC}$ cut BD in E;

therefore (Supp. xlii. 1.) $AE = EC$;

therefore (E. xxxviii. 1.)

$\triangle\, ABE = \triangle\, CBE$; and $\triangle\, APE = \triangle\, CPE$;

therefore $\triangle\, APB = \triangle\, CPB$;

therefore (E. xli. 1.) $\overline{AB} \times \overline{PK} = \overline{BC} \times \overline{PL}$;

therefore (E. xvi. 6.) $AB : BC :: PL : PK$.

Again, since (*constr.* E. xv. 1. E. xxxii. 1.) the $\triangle$s BKP, BNQ, are equiangular, as are, also, the $\triangle$s BLP, BMQ;

therefore (E. iv. 6.) $QM : BQ :: PL : PB$;

and $BQ : QN :: PB : PK$;

therefore (E. xxii. 5.) $QM : QN :: PL : PK$:

And it has been proved that $AB : BC :: PL : PK$;

therefore (E. xi. 5.) $AB : BC :: QN : QM$.

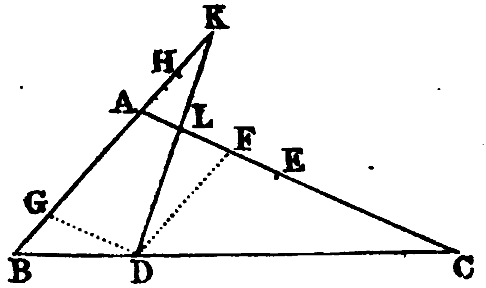
In the same manner, also, the proposition may be shewn to be true, if perpendiculars be let fall from Q on the sides DA, and DC, produced.

PROP. LIV.

63. PROBLEM. *From a given point, in the base of a scalene triangle, to draw a straight line, which shall cut off equal segments from the two remaining sides, the less of those sides having been produced.*

Let D be a given point in the base BC of the scalene $\triangle\, ABC$: It is required to draw from D a straight line which shall cut off from the greater side AC, and from the less side AB, produced, equal segments.

From AC cut off (E. iii. 1.) $CE = AB$; through



D draw (E. xxxi. 1.) DF parallel to AB, and DG parallel to AC; and, accordingly as the point F falls between E and C, or between E and A, take in AG, or in GA produced, $AH = EF$; produce HG, or GH, (Supp. lxxiii. 3. *Cor.*) to K, so that $\overline{GK} \times \overline{KH} = \overline{GA} \times \overline{AF}$; lastly, join D, K: Then shall DK cut off from AC a segment CL equal to the segment BK, which it cuts off from AB produced.

For, since (*constr.*) $GK \times KH$ is equal to $\overline{GA} \times \overline{AF}$, or (*constr.* and E. xxxiv. 1.) to $\overline{DF} \times \overline{GD}$,

therefore (E. xvi. 6.) $GK : GD :: DF : KH$:

But (*constr.* E. xxix. 1. and Supp. xxvi. 1.) the two △s KGD, LFD, are equiangular;

therefore (E. iv. 6.) $GK : GD :: DF : FL$:

therefore (E. ix. 5.) $KH = FL$:

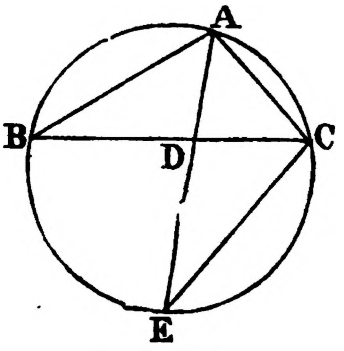
But (*constr.*) $BH = CF$; to these equals add the equals HK, and FL, and it is manifest, that the segment CL is equal to the segment BK.

PROP. LV.

64. THEOREM. *If an angle of a triangle be bisected by a straight line, which also cuts the base, the rectangle, contained by the sides of the triangle,*

is equal to the rectangle contained by the segments of the base, together with the square of the straight line bisecting the angle.

Let ABC be a triangle, and let the $\angle BAC$ be bisected by $\overline{AD}$; then $\overline{BA} \times \overline{AC} = \overline{BD} \times \overline{DC} + \overline{AD}^2$.



Describe (E. v. 4.) the circle ACB about the triangle; produce AD to the circumference in E, and draw $\overline{EC}$. And, because (E. xxi. 3.) the $\angle ABC = \angle AEC$, and (*hyp.*) the $\angle BAD = \angle CAE$; therefore (E. xxxii. 1.) the $\triangle$s ABD, AEC, are equiangular;

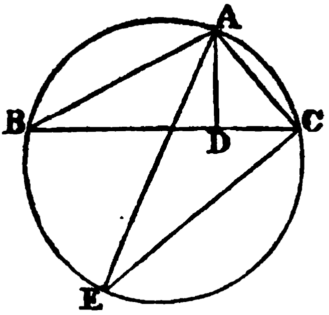
therefore (E. iv. 6.) $BA : AD :: EA : AC$;
therefore (E. xvi. 6.) $BA \times AC = \overline{EA} \times AD$;
that is, (E. iii. 2.) $BA \times AC = \overline{ED} \times \overline{DA} + AD^2$:
But (E. xxxv. 3.) $ED \times DA = BD \times DC$;
therefore $BA \times AC = BD \times DC + AD^2$.

PROP. LVI.

65. THEOREM. *If from any angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle, is equal to the rectangle contained by the perpendi-*

cular and the diameter of the circle described about the triangle.

Let ABC be a triangle, and $\overline{AD}$ the perpendicular



from the $\angle BAC$ to the base BC; then is $\overline{BA} \times \overline{AC}$ equal to the rectangle contained by AD, and the diameter of the circle described about the $\triangle ABC$.

Describe (E. v. 4.) the circle ACB about the triangle; draw its diameter AE, and join E, C: Because the $\angle ECA$ in a semi-circle is equal (E. xxxi. 3.) to the right $\angle BDA$, and that (E. xxi. 3.) the $\angle AEC = \angle ABC$; therefore (E. xxxii. 1.) the $\triangle$s ABD, AEC are equiangular,

therefore (E. iv. 6.) $BA : AD :: EA : AC$;
therefore (E. xvi. 6.) $BA \times AC = \overline{EA} \times AD$.

Prop. LVII.

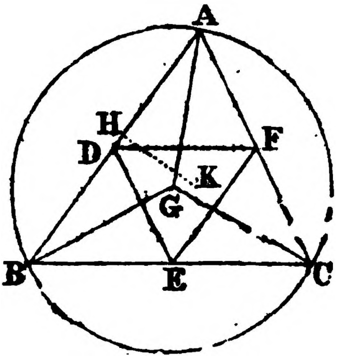
66. Theorem. *The rectangle contained by the diagonals of a quadrilateral rectilineal figure, inscribed in a circle, is equal to both the rectangles contained by its opposite sides.*

This is Prop. D. Book VI. of Euclid's Elements, as edited by Simpson.

Prop. LVIII.

67. Theorem. *If, from the centre of the circle, described about a given triangle, perpendiculars be drawn to the three sides, their aggregate shall be equal to the radius of the circumscribed circle, together with the radius of the circle inscribed in the given triangle.*

Let ABC be the given triangle; bisect (E. x. 1.) AB, BC, and AC in the points D, E, and F; and from D, E, and F draw (E. xi. 1.) DG perpendicular to AB, EG perpendicular to BC, and FG perpendicular to AC; then (Supp. iv. 1.) these perpendiculars meet in the same point G, which is the centre of the circle that can be described about the △ ABC; find, also, (E. iv. 4.) the centre K, and the semi-diameter KH, of the circle that can be inscribed in the △ ABC; and draw GA: Then *



$$GD+GE+GF=GA+KH.$$

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* The straight lines GD, GE, GF, are to be supplied in the figure.

For draw $\overline{DE}$, $\overline{EF}$, and $\overline{FD}$; therefore (Supp. lxix. 1. *Cor.* 1. and E. xxxiv. 1.) $\frac{1}{2}AC$, $CF=\frac{1}{2}AB$, and $FD=\frac{1}{2}BC$; draw $\overline{GB}$, and $\overline{GC}$; And, since (*constr.*) the angles at D, E, F, are right angles; therefore (E. xxxii. 1. *Cor.* 1.) the two $\angle$s DAF, DGF, are, together, equal to two right angles; therefore (Supp. xxviii. 3.) a circle may be described about the trapezium $ADGF$; and in the same manner it may be shewn that circles may be described about $BDGE$, and $CFGE$:
therefore (Supp. lvii. 6.) $\overline{AG}\times\overline{DF}+\overline{BG}\times DE+\overline{CG}\times\overline{FE}=\overline{AF}\times\overline{DG}+\overline{AD}\times\overline{GF}+\overline{BD}\times\overline{GE}+\overline{BE}\times\overline{DG}+\overline{CE}\times\overline{GF}+\overline{CF}\times\overline{GE}$: And if to the doubles of these equals be added the rectangles

$$GE\times\overline{BC}+GF\times\overline{AC}+\overline{GD}\times\overline{AB},$$

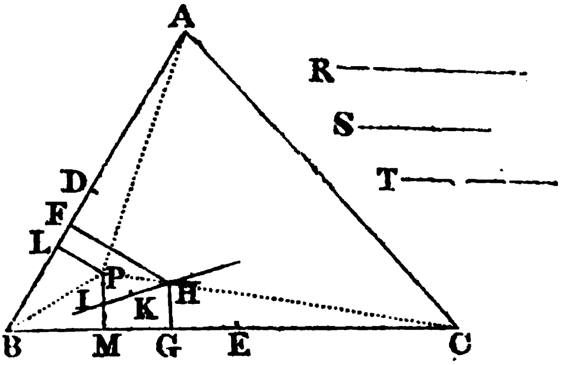
which (E. xli. 1.) make up the double of the $\triangle$ ABC, it will be manifest, from E. i. 2., that the rectangle contained by the perimeter of the $\triangle$ ABC, and by GA, together with the double of the $\triangle$ ABC, is equal to the rectangle contained by the perimeter of ABC, and by the aggregate of GD, GE, and GF: But (Supp. ii. 4.) the double of the $\triangle$ ABC is equal to the rectangle contained by the perimeter of the triangle and the semi-diameter, KH, of the circle inscribed in it; therefore (E. i. 2.) the rectangle contained by the perimeter, and by the aggregate of GA and KH, is equal the rectangle contained by the perimeter, and by the aggregate of GD, GE, and GF;

therefore $GD+GE+GF=GA+KH$.

Prop. LIX.

68. Problem. *To find a point, from which if three straight lines be drawn to three given points, their differences shall be severally equal to three given straight lines; the difference of any two of the straight lines to be drawn, not being greater than the distance of the two points to which they are to be drawn.*

Let *A*, *B*, *C*, be the three given points, and *R*, *S*, two of the given differences: It is required to find a point, from which if three straight lines be drawn to *A*, *B*, and *C*, the difference of the first and second shall be equal to *R*, the difference between the second and third equal to *S*, and therefore the difference between the first and third equal to the third of the given differences.



Draw $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$; bisect (E. x. 1.) *AB* in *D*, and *BC* in *E*; from *DB* cut off *DF*, equal to a third proportional (E. xi. 6.) to 2*AB*, and to *S*; likewise from *EB* cut off *EG*, equal to a third proportional to 2*BC*, and to *R*; and through *F* and *G* draw (E. xi. 1.) *FH* perpendicular to *AB*, and *GH* perpendicular to *BC*, and let them meet in *H*; find (E. xii. 6.) a fourth proportional, (*T*), to $\overline{AB}$, *S*, and *BC*; through *H* draw (Supp. vii. 6.) the *locus*, *IH*, of all the points, from which if perpendiculars be drawn to *AB* and *BC*, respectively, they shall cut off from *GB* and *FB* segments that are to one another as *R* is to *T*; lastly, in *IH* find (Supp. xcvi. 3.) a point *K*, such that the difference of its distances from *C* and *B*, shall be equal to *R*: Then is *K* the point which was to be found.

For if not, let *P* be the point; and, if it be possible, let the point *P* be out of *IH*; join *P*, *A*, and *P*, *B*, and *P*, *C*; and draw, from *P* (E. xii. 1.) *PL* perpendicular to *AB*, and *PM* perpendicular to *BC*: Then (*constr*. E. xvii. 3. and Supp. xcvi. 3. *Cor*. 1.) $\overline{FL} \times AB = \overline{BP} \times \overline{S}$; and $\overline{GM} \times BC = \overline{BP} \times \overline{R}$; therefore (E. xvi. 6. and *constr*.)

$$BP : FL :: AB : S :: BC : T;$$

$$\text{and } GM : BP :: R : BC;$$

therefore (E. xxiii. 5.) $GM : FL :: R : T$: therefore (*constr*. and Supp. vii. 6. *Cor*.) the point *P* cannot be out of *IH*; therefore (*constr*. and Supp. xcvi. 3. *Cor*. 2.) *K* is the point which was to be found.

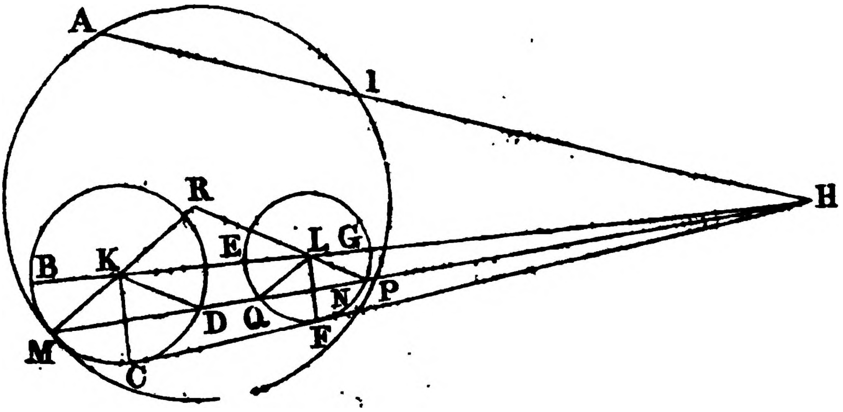
PROP. LX.

69. **PROBLEM.** *To describe a circle, which shall pass through a given point, and touch two given circles.*

Find a point (Supp. lix. 6.) such that the difference between its distance from the centre of the one circle, and its distance from the given point, shall be equal to the semi-diameter of that circle; and that the difference between its distance from the other centre, and from the given point, shall likewise be equal to the other given semi-diameter: It is manifest (Supp. vi. 3.) that the point, so determined, is the point which was to be found.

OTHERWISE.

Let *BCD* and *EFG* be the two given circles, and



A a point without them: It is required to describe a circle which shall pass through *A*, and touch the two circles *BCD*, *EFG*.

Find (E. i. 3.) the centres, *K* and *L*, of the two given circles; draw $\overline{KL}$ and let it, produced, meet the circumference of *BCD* in *B*, and the circumference of *EFG* in *E* and *G*; and let it meet $\overline{CH}$, which is drawn (Supp. lii. 3.) so as to touch both the circles, in *H*; join *H*, *A*; find (E. xii. 6.) a fourth proportional to *AH*, *HB*, and *HG*, and from *HA* cut off *HI* equal to it; so that (E. xvi. 6.)

$$\overline{BH}\times\overline{HG}=\overline{AH}\times\overline{HI};$$

lastly, describe (Supp. xcv. 3.) a circle *AMI*, passing through *A* and *I*, and touching either of the circles *BCD*, in some point, *M*; it shall, also, if $\overline{HM}$ be drawn, pass through the point *N*, in which *HM* cuts the circumference of the circle *EFG*, and shall touch *EFG* in the point *N*.

For, if it be possible, let the circumference of the circle *AMI* cut *HM* in some other point, as *P*: Then (E. xxxvi. 3.)

$$\overline{MH} \times \overline{HP} = AH \times HI;$$

$$\text{but } (\textit{constr.})\ AH \times HI = BH \times HG,$$

$$\text{and (E. xxxvi. 3.) } BH \times HG = MH \times \overline{HN};$$

$$\therefore MH \times HP = MH \times HN;$$

∴ *HP* is equal to *HN*, the less to the greater, which is absurd; therefore the circumference of the circle *MIA* cannot but meet the circle *EFG* in the point where it is cut by *MH*; and it touches the circle *EFG* in that point.

For draw *KM*, *KC*, *KD*, *LQ*, *LF*, and *LP*, and let *MK* and *PL*, produced, meet in *R*: Then since (*constr.* and E. xviii. 3.) the △s *HFL*, *HCK*, having a common angle at *H*, have the ∠s *HFL*, *HCK*, right angles, they are (Supp. xxvi. 1.) equiangular; therefore (E. iv. 6.)

$$HL : LF \text{ or } LQ :: HK : KC \text{ or } KM:$$

And the △s *HLQ*, *HKM*, have a common angle at *H*, and have the two remaining ∠s *HQL*, *HMK* of the same species; for since *MH* cuts both the circles, the ∠s *HQL*, *HMK*, are (E. xvi. 3. *Cor.*) each of them less than a right angle; therefore (E. vii. 6.) the ∠ *HQL* = ∠ *HMK*; therefore (E. xv. Def. 1. and E. v. 1.) the ∠ *LNQ*, or *RNM*, equal ∠ *HMK*, or *NMR*; therefore (E. vi. 1.) *RM* = *RN*; but, since the circle *AMI* touches the circle *BCD*, of which *K* is the centre; therefore (E. xi. or xii. 3.) the centre of *AMI* must be in *MR*; and since *RM* $=$ *RN*, that centre (E. vii. 3.) must be in *R*; since, therefore, the diameters of the two circles *MAI*, *EFG*, have a common extremity at *N*, the two circles (Supp. vi. 3.) touch one another*.

Prop. LXI.

70. Problem. *To describe a circle that shall touch three given circles.*

Find a point (Supp. lix. 6.) such that the difference between its distances from the centres of the first and second of the given circles, shall be equal to the difference of the diameters of those circles, and such that the difference between its distances from the centres of the first and third of the given circles, shall be equal to the difference of the diameters of those circles: Then it is manifest, that the difference between its distances from the centres of the second and third of the given circles, will be equal to the difference of their diameters; and that, if from the point so determined, as a centre, a circle be described touching any one of the given circles, it will (Supp. iv. 3.) also touch the other two.

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* It is evident that Prop. lix. may be deduced from this proposition, as it is thus independently demonstrated; and that the proposition immediately following, which is one of some celebrity, may be deduced from either of them.

PROP. LXII.

71. PROBLEM. *Upon a given finite straight line, to describe an equilateral and equiangular figure having the number of its sides equal to four, eight, sixteen, &c.; or to three, six, twelve, &c.; or to five, ten, twenty, &c.; or to fifteen, thirty, sixty, &c. sides.*

In any circle inscribe (Supp. xiv. 4. *Cor.* 3.) an equilateral and equiangular rectilineal figure of any number of sides that is specified in the proposition; then upon the given finite straight line describe (E. xviii. 6.) a rectilineal figure similar to it, and the problem will have been solved.

PROP. LXIII.

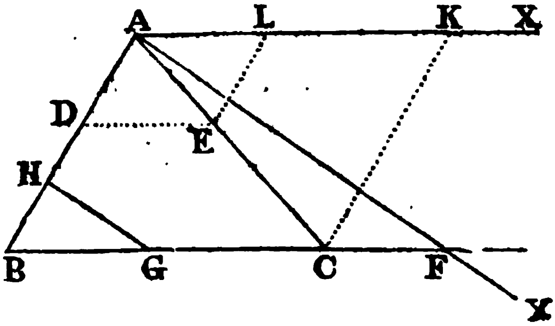
72. THEOREM. *Similar triangles, and similar polygons, are to one another as any rectilineal figure described upon any side of the one, is to a similar rectilineal figure similarly described upon the homologous side of the other.*

For (E. xx. 6.) the two given figures, and two similar figures thus similarly described, will have to one another the same duplicate ratio of that which the homologous sides have.

PROP. LXIV.

73. PROBLEM. *To cut off from a given triangle any part required, by a straight line drawn parallel to a given straight line.*

Let *ABC* be the given triangle, and *AX* a given



straight line: It is required to cut off, from the $\triangle ABC$, any assigned part, by a straight line drawn parallel to *AX*.

First, let *AX* be parallel to *BC*; find (Supp. xxi. 6.) a square which shall be the same part of the square of *AB*, that the triangle, to be cut off, is required to be of the given triangle, and make *AD* equal to its side; through *D* draw (E. xxxi. 1.) *DE* parallel to *AX* or *BC*: Then is *ADE* the triangle which was to be cut off from *ABC*.

For, (*constr.* and E. xxix. 1.) the $\triangle$s *ADE*, *ABC*, are equiangular;

therefore (E. iv. 6. and Supp. lxiii. 6.)

$$\overline{AD}^2 : \overline{AB}^2 :: \triangle ADE : \triangle ABC.$$

Therefore (*constr.* and Supp. iv. 5.) the $\triangle$ *ADE* is the required part of the $\triangle$ *ABC*.

Secondly, let *AX* be not parallel to *BC*, and let it meet *BC*, produced, if necessary, in *F*: Find (Supp. xxi. 6.) a square which shall be the same part of the rectangle $\overline{FB} \times \overline{BC}$, that the triangle, to be cut off, is required to be of the $\triangle$ *ABC*, and make *BG* equal to its side; through *G* draw *GH* parallel to *FA*: Then is *BHG* the triangle which was to be cut off from *ABC*.

For (*constr.* and E. xxix. 1.) the $\triangle$s *BHG*, *BAF*, are equiangular;

therefore (E. iv. 6. and Supp. lxiii. 6.)

$$\overline{BG}^2 : \overline{BF}^2 :: \triangle BHG : \triangle BAF;$$

And (E. i. 6. and E. xi. 5.)

$$BF^2 : BF \times BC :: \triangle BAF : \triangle BAC;$$

therefore (E. xxii. 5.)

$$BG^2 : BF \times BC :: \triangle BHG : \triangle BAC;$$

therefore (*constr.* and Supp. iv. 5.) the $\triangle$ *BHG* is the required part of the $\triangle$ *ABC*.

PROP. LXV.

74. PROBLEM. *To describe a polygon, similar to a given polygon, and having a given ratio to it.*

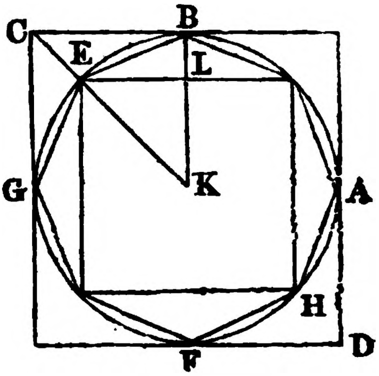
Upon any side of the given polygon describe (E. xlvi. 1.) a square; find (Supp. xxi. 6.) a square which shall have to the square first described the given ratio; and upon its side describe (E. xviii. 6.) a polygon similar, and similarly situated, to the given polygon: It is manifest, from E. xx. 6., that it will have to the given polygon the given ratio.

PROP. LXVI.

75. THEOREM. *Any regular polygon, inscribed in a circle, is a mean proportional between the inscribed and circumscribed regular polygons of half the number of sides.*

Let *BGFA* be a polygon inscribed in the circle

BG, and let *EH* and *CD* be polygons of half the



number of sides, the one *EH* inscribed in the circle (Supp. xiv. 4. *Cor.* 3.) by joining the sides of the figure *BGFA*, and the other *CD* described about the circle, by drawing tangents to it through the angular points *A*, *B*, *G*, and *F*; so that (E. xviii. 3. E. xxviii. 3. E. xxvii. 3. E. xxvi. 3. and Supp. xix. 3.) it is equilateral and equiangular: Then is the polygon *BGFA* a mean proportional between the polygons *EH* and *CD*.

Find (E. i. 3.) the centre *K* of the circle *BGFA*, and join *K*, *B*, and *K*, *C*: It is manifest, from the construction, that *KB* bisects, at right angles, the sides of the figures *EH* and *CD*, which it cuts, and that *KC* passes through the angular point *E*:

And (E. i. 6. and E. iv. 6.)

$$\triangle CBK : \triangle EBK :: CK : EK :: CB : EL:$$

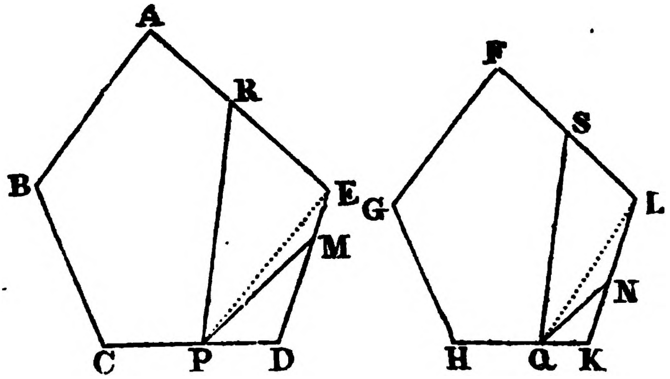
But, the figure *CD* is the same multiple of the $\triangle CBK$, that the figure *BGFA* is of the $\triangle EBK$; also a side of *CD* is double of *CB*; and a side of *EH* is double of *EL*; therefore (E. xv. 5.) *CD* is to *BGFA* as a side of *CD* is to a side of *EH*; and (E. xx. 6.) *CD* has to *EH* the duplicate ratio, of that which a side of *CD* has to a side of *EH*; therefore *CD* has to *EH* the duplicate ratio, of that which it has to *BGFA*; that is, (E. x. Def. 5.) the figure *BGFA* is a mean proportional between *CD* and *EH*.

PROP. LXVII.

76. THEOREM. *If from two points similarly situated, one in each of any two homologous sides of two similar polygons, two straight lines be drawn making equal angles with those sides, they shall cut off from the polygons two similar figures; and the one shall be the same part of the one polygon, that the other is of the other.*

Let *AC* and *FH* be two similar polygons, and *P* and *Q* two points similarly situated in the two homologous sides *CD* and *HK*: If from *P* and *Q* straight lines be drawn, making equal angles with *CD* and *HK*, they shall cut off similar figures from the polygons; and the one shall be the same part of the one polygon that the other is of the other.

First, let *PM* and *QN*, making the $\angle MPD = \angle NQK$, cut the sides *DE* and *KL*, adjacent to *CD* and *HK*: And since (*hyp.* and Supp. xxvi. 1.) the two $\triangle$s *MPD*, *NQK*, are equiangular, they are (E. iv. 6.) similar to one another, and they are to one another (E. xix. 6.) in the duplicate ratio of their homologous sides *PD* and *QK*, that is (*hyp.*) in the duplicate ratio of *CD* and *HK*; therefore (E. xx. 6.) they are to one another in the same ratio as the polygons are, and therefore whatever part the



△*MPD* is of the polygon *ABCDE*, the same part is the △*NQK* of the polygon *FGHKL*.

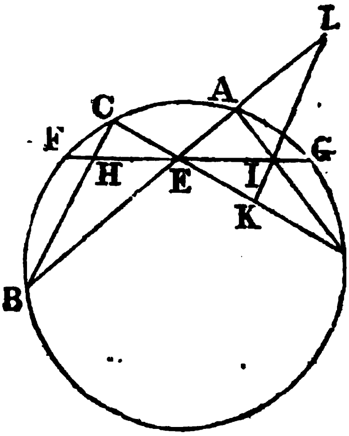
Secondly, let *PR* and *QS* cut any other sides of the polygons, as *AE* and *FL*, which are not adjacent to the sides *CD* and *HK*: Draw *PE* and *QL*; and since (*hyp.* and E. xxvi. 1.) the △s *EPD* and *LQK* are equiangular, the △s *RPE* and *SQL* are also equiangular; whence it may be shewn, (as in E. xx. 6.) that *RPDE*, *SQKL*, are similar figures; therefore (E. xx. 6.) they are to one another in the duplicate ratio of the homologous sides *DE* and *KL*; or in the ratio of the polygon *ABCDE* to the polygon *FGHKL*; ∴ *RPDE* is the same part of *ABCDE* that *SQKL* is of *FGHKL*.

PROP. LXVIII.

77. THEOREM. *If any two chords of a circle intersect each other, the straight lines joining their extremities shall cut off equal segments from the*

chord which passes through the common intersection of the two former chords and is there bisected.

Let AB and CD be two chords of the circle



$ACBD$, cutting one another in E; through E draw (Supp. ii. 3.) the chord FG, so that FG is bisected in E; and join C, B and A, D: Then shall $HE = EI$.

For through I draw (E. xxxi. 1.) KIL parallel to BC, and meeting CD in K, and BA, produced, in L: Then (*constr*. and E. xxix. 1.) the $\angle CBL = \angle BLK$, and that (E. xxi. 3.) the $\angle CBA = \angle CDA$, therefore the $\angle ALI = \angle IDK$; and (E. xv. 1.) the $\angle AIL$, of the $\triangle LAI$, is equal to the $\angle KID$, of the $\triangle DKI$; therefore (Supp. xxvi. 1.) these two triangles are equiangular, as are also (*constr.* and E. xxix. 1.) the two $\triangle$s CEH, IEK, and the two $\triangle$s HEB, IEL;

therefore (E. iv. 6.) $AI : IL :: KI : ID$;

therefore (E. xvi. 6.) $\overline{IL} \times \overline{KI} = \overline{AI} \times \overline{ID}$:

Again (E. iv. 6.) $CH : HE :: IK : IE$,

and $BH : HE :: IL : IE$,

therefore (E. xxii. 6.)

$\overline{CH} \times \overline{BH} : \overline{HE}^2 :: \overline{IK} \times IL$ or $AI \times ID : \overline{IE}^2$;

therefore (E. xviii. 5.)

$CH \times BH + HE^2 : HE^2 :: AI \times ID + IE^2 : IE^2$;

therefore (E. xxxv. 3.)

$\overline{FH} \times \overline{HG} + HE^2 : HE^2 :: \overline{FI} \times IG + IE^2 : IE^2$:

But (*hyp.* and E. v. 2.) $FH \times HG + HE^2$, and $FI \times \overline{IG} + IE^2$, are each of them equal to $\overline{EF}^2$ or $\overline{EG}^2$, and therefore they are equal to one another; therefore (E. xiv. 5.) $HE^2 = IE^2$; $\therefore HE = IE$.

PROP. LXIX.

78. PROBLEM. *Two similar rectilineal figures being given, to find a third figure also similar to them and a mean proportional between them.*

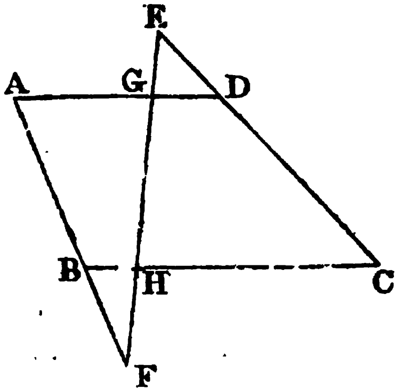
Find (E. xiii. 6.) a mean proportional between any two homologous sides of the given figures, and upon it describe (E. xviii. 6.) a rectilineal figure similar to either of them, and therefore (E. xxi. 6.) similar, also, to the other: Then (E. xxii. 6.) will the rectilineal figure, so described, be a mean proportional between the two given figures.

PROP. LXX.

79. PROBLEM. *If two sides of a trapezium be parallel, and a straight line be drawn cutting them, and meeting also the other two sides, (any of the sides being produced, if necessary) the two rectangles contained by the respective segments of the parallel sides, have to each other the same ratio, as*

the two rectangles contained by the segments into which the line, so drawn, is severally divided by each of the two parallels.

Let the side *AD*, of the trapezium *ABCD*, be



parallel to the opposite side *BC*, and let *EF* cut *AD* and *BC*, in *G* and *H*, and *AB* and *DC*, produced, in *F* and *E*: Then

$$\overline{AG}\times\overline{GD} : \overline{BH}\times\overline{HC} :: \overline{GF}\times\overline{EG} : \overline{HF}\times\overline{EH}.$$

For (*hyp.* and E. xxix. 1.) the △s *AGF*, *BHF*, and the △s *EGD*, *EHC*, are equiangular; therefore (E. iv. 6.)

$$AG : BH :: GF : HF;$$

$$\text{and } GD : HC :: EG : EH;$$

therefore (Supp. i. 6.)

$$AG\times GD : BH\times HC :: GF\times EG : HF\times EH.$$

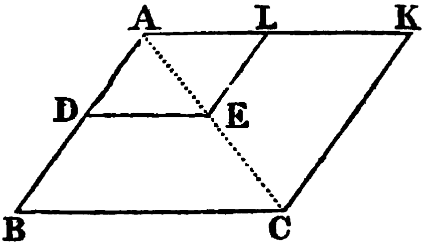
Which conclusion may also be arrived at by means of E. xxiii. 6.

PROP. LXXI.

80. PROBLEM. *To cut off from a given parallelo-*

gram a similar parallelogram, which shall be any required part of it.

Let *ABCK* be the given parallelogram: It is



required to cut off from it a similar parallelogram, which shall be any required part of it.

Draw the diameter *AC*, and from the △*ABC* cut off (Supp. lxiv. 6.) by a straight line *DE*, drawn parallel to *AK*, the △*ADE* the same part of *ABC* as the parallelogram to be cut off is required to be of the given parallelogram; through *E* draw (E. xxxi. 1.) *EL* parallel to *AB*: Then, since (E. xxxiv. 1.) the ▭*ADEL* is the double of the △*ADE*, it is (*constr.* and E. xv. 5.) the required part of the ▭*ABCK*; and (E. xxiv. 6.) the ▭*ADEL* is, also, similar to the ▭*ABCK*.

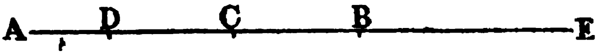
81. Cor. Hence, a gnomon may be cut off from a given parallelogram, which shall be any required part of it.

Prop. LXXII.

82. Theorem. *A given straight line being cut in extreme and mean ratio, if from the greater segment the less be taken, the greater segment also will thus be cut in extreme and mean ratio; and if a straight line, equal to the greater segment, be*

added to the given line, the line which is made up of the given line and this segment, is also cut in extreme and mean ratio.

Let AB be a finite straight line; let it be



cut (E. xxx. 6.) in extreme and mean ratio in the point C; from the greater segment, AC, cut off $CD = CB$; and to AB add $BE = AC$: Then shall AC be cut in extreme and mean ratio in the point D; and AE shall be cut in extreme and mean ratio in the point B.

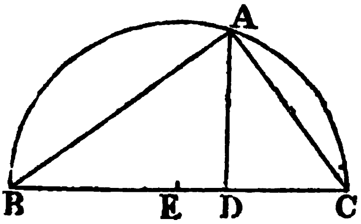
For since (*hyp.*) $AB : AC :: AC : CB$ or CD, therefore (E. xvii. 5.) CB or $CD : AC :: AD : CD$; therefore (Supp. ii. 5.) $AC : CD :: CD : AD$; therefore (E. iii. Def. 6.) AC is cut in extreme and mean ratio in the point D.

Again, since
(*hyp.* and Supp. ii. 5.) AC or $BE : AB :: CB : AC$, therefore (E. xviii. 5.) $AE : AB :: AB : AC$ or BE; therefore (E. iii. Def. 6.) AE is cut in extreme and mean ratio in the point B.

PROP. LXXIII.

83. PROBLEM. *Upon a given straight line, as an hypotenuse, to describe a right-angled triangle, which shall have its three sides continual proportionals.*

Let BC be the given finite straight line: It is required to describe upon it a right-angled triangle,



the sides of which shall be continual proportionals.

Cut (E. xxx. 6.) *BC* in extreme and mean ratio, in the point *D*; bisect (E. x. 1.) *BC* in *E*; from the centre *E*, at the distance *EB* or *EC*, describe the circle *BAC*, and let *DA*, drawn from *D*, (E. xi. 1.) perpendicular to *BC*, meet its circumference in *A*, and join *A*, *B* and *A*, *C*: Then is *ABC* the triangle which was to be described.

For (*constr*. and E. xxxi. 3.) the $\angle BAC$ is a right angle; therefore (*constr*. and E. viii. 6. *Cor*.) *AC* is a mean proportional between *BC* and *DC*, as is also (*constr*.) *BD*; $\therefore AC = BD$; therefore but (E. viii. 6. *Cor*.) *AB* is a mean proportional between *BC* and *BD*; therefore *AB* is a mean proportional between *BC* and *AC*.

84. Cor. 1. *CB*, *BA*, *AC*, and *AD*, are continual proportionals.

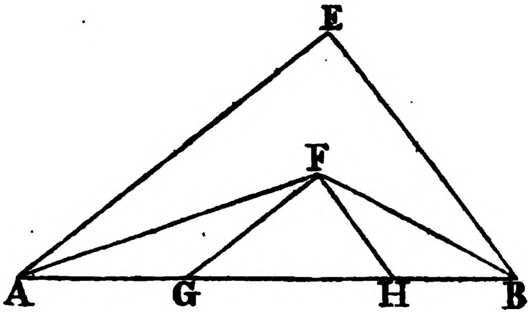
For (E. viii. 6.) $AB : BD :: AC : AD$; and it has been shewn that $AC = BD$.

85. Cor. 2. It is manifest, from the demonstration, that if the three sides of a right-angled triangle be continual proportionals, the hypotenuse will be divided in extreme and mean ratio, by a perpendicular drawn to it from the right angle.

PROP. LXXIV.

86. PROBLEM. *The perimeter being given of a right-angled triangle, having its three sides proportionals, to construct the triangle.*

Let *AB* be a straight line: It is required to describe a right-angled triangle, which shall have its sides continual proportionals, and equal together to *AB*.



Upon *AB* describe (Supp. lxxiii. 6.) the right-angled △ *AEB*, having its sides continual proportionals; bisect (E. ix. 1.) the ∠s *EAB*, *EBA*, by two straight lines *AF* and *BF*, which meet in *F*; and through *F* draw (E. xxxi. 1.) *FG* parallel to *EA*, and *FH* parallel to *EB*: Then is *FGH* the triangle which was to be described.

For it may be shewn, as in Supp. xxxiv. 1., that the perimeter of the △ *FGH* is equal to *AB*; and since (*constr.* E. xxix. 1. and Supp. xxvi. 1.) the △s *FGH* and *EAB* are equiangular, and that the sides of the △ *EAB* are proportionals, it is manifest from E. iv. 6., and E. xi. 5., that the sides of the △ *FGH* will, also, be proportionals.

PROP. LXXV.

87. THEOREM. *The semi-diameter of a given circle having been divided in extreme and mean ratio, the greatest segment shall be equal to the side of an equilateral and equiangular decagon inscribed in the circle.*

For (Supp. xiv. 4. *Cor.* 1.) if the semi-diameter of the circle be so divided, as that the rectangle, contained by the whole and the lesser part, may be equal to the square of the greater part, that greater segment will be equal to the side of an equilateral and equiangular decagon to be inscribed in the given circle; and (E. xvii. 6. E. iii. Def. 6.) when the semi-diameter has been so divided, it is cut in extreme and mean ratio.

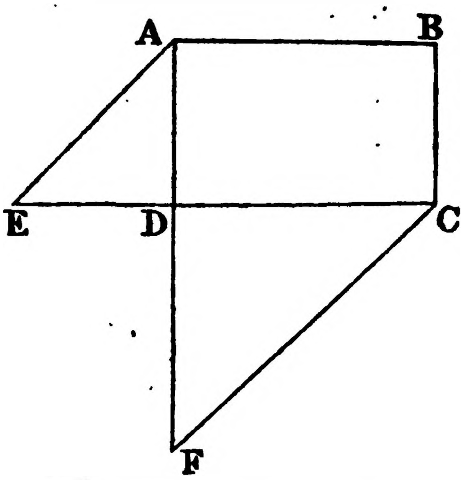
PROP. LXXVI.

88. THEOREM. *Any rectangle is the half of the rectangle contained by the diameters of the squares of its two sides.*

Let $ABCD$ be a rectangle; produce AD to F, and make $DF = DC$; produce, also, CD to E, and make $DE = DA$; join A, E and C, F; therefore AE and CF are the diameters of the squares of AD and DC: Then is the rectangle $ABCD$ equal to the half of $\overline{AE} \times \overline{CF}$.

For (*constr.* E. v. 1. and E. xxxii. 1.) the two △s ADE, CDF, are equiangular;
therefore (E. iv. 6.) $CF : AE :: CD : DE$ or DA;
therefore the rectangle contained by CF and AE is

(E. 1. Def. 6.) similar to the rectangle contained by



CD and *DA*; and since (*hyp.* and E. x. Def. 1.) the ∠ *FDC* is a right angle, therefore the rectangle $CF \times AE$, which is on *CF*, is equal (E. xxxi. 6.) to the two similar rectangles $\overline{CD} \times \overline{DA}$ and $\overline{FD} \times \overline{DE}$, which are on the equal sides *CD* and *DF*; that is, the rectangle $CF \times AE$ is double of the rectangle $CD \times DA$; or this latter rectangle is equal to the half of the former.

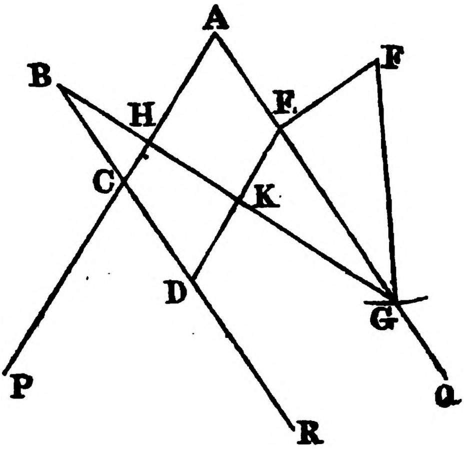
PROP. LXXVII.

89. PROBLEM. *Through a given point, to draw a straight line, cutting two given straight lines, which meet one another, so that the triangle contained by the segment of that line and the segments which it cuts off from the given lines, shall be equal to a given rectilineal figure.*

Let *AP* and *AQ* be two given straight lines, which meet in *A*, and, first, let *B* be a given point without the ∠ *PAQ*: It is required to draw through

B a straight line cutting *AP* and *AQ*, so that the triangle, contained by the segments of the three lines, shall be equal to a given rectilineal figure.

Through *B* draw (E. xxxi. 1.) *BR* parallel to *AQ*, and let it meet *AP* in *C*; to *AC* apply (E. xlv. 1. *Cor.*) the ▱ *ACDE*, having the ∠ *ACD* for one of its angles, and equal to the given figure; from *E* draw (E. xi. 1.) *EF* perpendicular to *AQ*, and make it equal to *BC*; from the centre *F*, at a distance equal to *BD*, describe a circle cutting *AQ* in *G*, and join *F*, *G*; ∴ $FG = BD$; lastly draw *BG*, cutting *AP* in *H*: Then is *BG* the line which was to be drawn.



For, let *BG* cut *ED* in *K*; the △s *BCH*, *BDK*, *EKG*, are (*constr.* E. xxix. 1. and E. xv. 1.) equiangular; and therefore (E. iv. 6.) similar to one another; and (*constr.*) $BC = EF$, $BD = FG$, and the ∠ *FEG* is a right angle; it is manifest, therefore, that the three homologous sides *BC*, *BD*, and *EG*, of the three similar △s *BCH*, *BDK*, *EKG*, are the sides of a right-angled triangle; therefore

(E. xxxi. 6.) the △*BDK* = △*BCH* + △*EKG*; take away the △*BCH*, and there remains the trapezium *HCDK* = △*EKG*; add to both these the trapezium *AHKE*, and it is evident that the △*AHG* is equal to the ▭*ACDE*, or (*constr.*) to the given figure.

And, in a similar manner may the problem be solved, when the given point is within the rectilineal ∠*PAQ*, in all cases in which the problem so restricted admits of a solution.

90. Cor. Hence, and from Supp. xxi. 6., a straight line may be drawn through a given point, which shall cut off from a given triangle any required part of it *.

Prop. LXXVIII.

91. Problem. *Through a given point to draw a straight line, so as to cut off from two straight lines, that meet one another, two segments, toward their point of concourse, which shall contain a rectangle equal to a given square.*

Let the two given straight lines *AP* and *AQ* meet in *A*; and let *B* be a given point either within or without the ∠*PAQ*: It is required to draw through *B* a straight line, so as to cut off from *AP* and *AQ* two segments, towards *A*, which shall contain a rectangle equal to a given square.

From *AP* and *AQ* cut off *AC* and *AD* each of them equal to a side of the given square, and join

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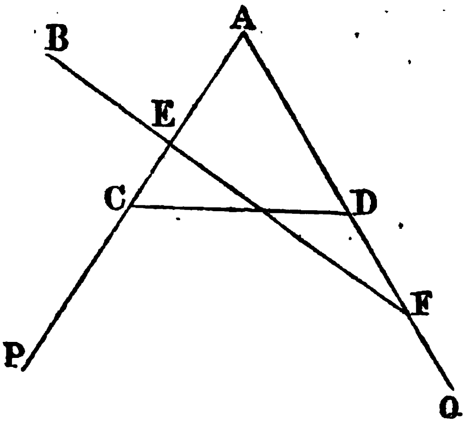
* Hence, and by the help of Trigonometry, any given rectilineal figure may be divided into two parts, which are to each other in any given ratio, by a straight line drawn from a given point, situated without the given figure.



C, *D*; through *B* draw (Supp. lxxvii. 6.) $\overline{BEF}$ cutting off the $\triangle AEF$, equal to the $\triangle ACD$: Then, since the two $\triangle$s *EAF*, *CAD*, have the same vertical angle, therefore (Supp. v. 6. *Cor.*)

$\triangle EAF : \triangle CAD :: \overline{EA}\times\overline{AF} : \overline{CA}\times\overline{AD}$ or $\overline{AC}^2$;

But (*constr.*) the $\triangle EAF = \triangle ACD$;

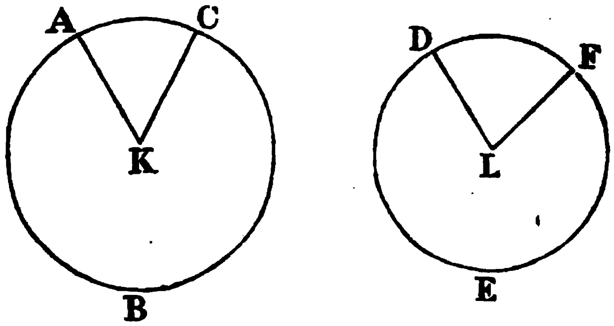
$\therefore EA\times AF = AC^2$; that is, (*constr.*) the rectangle $EA\times AF$ is equal to the given square.

PROP. LXXIX.

92. THEOREM. *In different circles the semidiameters which bound equal sectors contain angles reciprocally proportional to their circles; and conversely.*

In the circles *ABC*, *DEF*, of which *ABC* is the greater, first, let *AKC*, *DLF*, be two equal sectors: Then shall the $\angle AKC$ be to the $\angle DLF$, as the circle *DEF* is to the circle *ABC*.

For (E. xxxiii. 6.)
$\angle AKC$: four right angles :: sector AKC : circle ABC;
and,
four right angles : $\angle DLF$ :: circle DEF : sector DLF:



But (*hyp.*) sector $AKC =$ sector DLF;
therefore (E. xxxiii. 5.)
$\angle AKC$: $\angle DLF$ :: circle DEF : circle ABC.

Secondly, let the $\angle AKC$ be to the $\angle DLF$, as the circle DEF is to the circle ABC: Then shall the sector AKC be equal to the sector DLF.

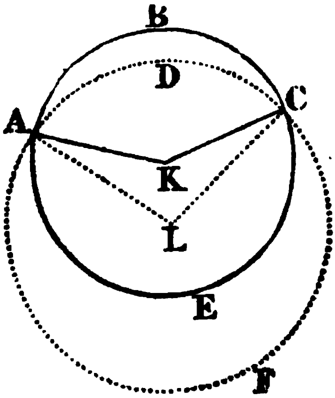
For it may be shewn as before that
four right angles : $\angle DLF$ :: circle DEF : sector DLF;
and (*hyp.* and Supp. iii. 5.)
$\angle DLF$: $\angle AKC$ :: circle ABC : circle DEF;
and
$\angle AKC$: four right angles :: sector AKC : circle ABC;
therefore (E. xxiii. 5.) four right angles : four right angles :: sector AKC : sector DLF;
therefore sector $AKC =$ sector DLF.

Prop. LXXX.

93. **Problem.** *A given space being bounded by two arches of circles, subtending, at their respective*

centres, angles reciprocally proportional to the circles, to find a square that shall be equal to it.

Let the space *ABCDA* be bounded by arches



ABC, ADC, of circles *ABCE, ADCF,* the centres of which are *K* and *L*; let the ∠*AKC* be to the ∠*ALC,* as the circle *ADCF* is to the circle *ABCE*: It is required to find a square that shall be equal to the space *ABCDA.*

Since (*hyp.* and Supp. lxxix. 6.) the sector *ABCK* is equal to the sector *ADCL,* from these equals take away the common part *ADCK*; and there remains the figure *ABCDA* equal to the rectilineal figure *AKCL*: Find, therefore, (E. xiv. 2.) a square equal to the figure *AKCL,* and the problem will have been solved.

Prop. LXXXI.

94. **Problem.** *To trisect a given circle, by dividing it into three equal sectors.*

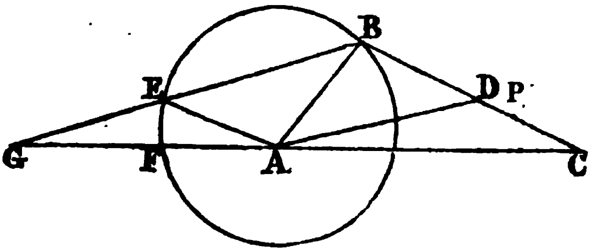
Inscribe (E. i. 1. E. ii. 4.) in the given circle an equilateral triangle: Then it is manifest, from E. xxviii. 3. and E. xxxiii. 6., that the straight lines drawn from the centre of the circle to the three angular points of the inscribed triangle, will divide the circle into three equal sectors.

95. Cor. In like manner, a circle may be divided into any required number of equal sectors, in all cases in which an equilateral figure, having that same number of sides, can be inscribed in the circle.

Prop. LXXXII.

96. Theorem. *If, from the greater of two unequal sides of a given triangle, be cut off a part equal to the less, that segment shall have to the remaining segment, a ratio greater than the ratio which the angle adjacent to the remaining segment, has to the angle adjacent to the segment first cut off.*

Let the side AB, of the triangle ABC, be less than the side BC, and from BC let there be cut off $BD = AB$: Then



$$(BD : DC) > (\angle ACB : \angle ABC).$$

For draw $\overline{AD}$, and complete (E. xxxi. 1.) the ▭ $ADBE$; from the centre A, at the distance AB, describe the circle BEF, the circumference of which, since (E. xxxiv. 1. and *constr.*) $AE=BD$ or AB, will pass through E; produce CA to meet the circumference BEF in F, and BE produced in G; so that (E. xxix. 1.) the △s GEA, GBC, are equiangular.

Then, since the sector AEF is less than the △ AEG, and the sector AEB greater than the △ AEB,

$\therefore$ (△AEG : sector AEF) > (△AEB : sector AEB); therefore (Supp. vii. 5.)

(△AEG : △AEB) > (sector AEF : sector AEB):

But (E. i. 6.) △AEG : △AEB :: GE : EB,

and (E. iv. 6. and E. xxxiv. 1.)

$$GE : EB :: BD : DC;$$

$\therefore$ (BD : DC) > (sector AEF : sector AEB).

Also (E. xxxiii. 6.)

sector AEF : sector AEB :: $\angle EAF$: $\angle EAB$;

and (E. xxix. 1.)

the $\angle EAF = \angle ACB$, and the $\angle EAB = \angle ABC$;

$\therefore$ sector AEF : sector AEB :: $\angle ACB$: $\angle ABC$;

$$\therefore (BD : DC) > (\angle ACB : \angle ABC).$$

97. COR. If any part BP be taken of BC, that is greater than AB, then, much more, is

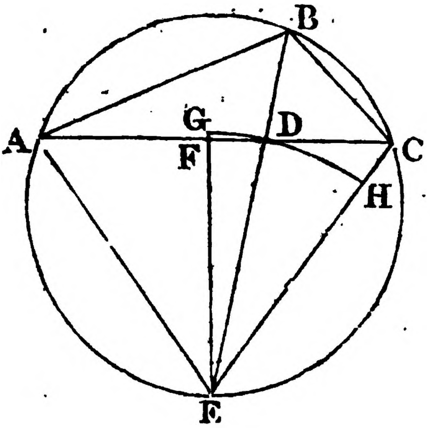
$$(BP : PC) > (\angle ACB : \angle ABC).$$

PROP. LXXXIII.

98. THEOREM. *The greater of any two unequal arches, of a given circle, has a greater ratio to the less arch, than the chord of the greater has to the chord of the less.*

Let $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ be any two unequal arches of the circle $AECB$, and let $\overset{\frown}{AB}$ be the greater: Then

$$(\overset{\frown}{AB} : \overset{\frown}{BC}) > (\overline{AB} : \overline{BC}).$$



For join A, C; bisect (E. ix. 1.) the $\angle ABC$ by BDE, and let BDE cut AC in D, and the circumference in E; join, also, E, C, and E, A; from E draw (E. xii. 1.) EF perpendicular to AC.

And, since (*constr.* and E. xxvi. 3.) $\overset{\frown}{CE} = \overset{\frown}{AE}$, therefore (E. xxix. 3.) $\overline{CE} = \overline{AE}$; and $\overline{EF}$ is common to the two right-angled $\triangle$s AFE, CFE; therefore (Supp. lxxiii. 1.) $\overline{AF} = \overline{FC}$, and therefore $\overline{AD} > \overline{DC}$; also (E. xix. 1.) $\overline{EC} > \overline{ED}$, and $ED > EF$; if, therefore, from the centre E, at the distance ED, a circle GDH be described, it will cut $\overline{EC}$, in H, between E and C, and EF, produced, in F; so that the sector $DEG > \triangle\, DEF$, and the sector $DEH < \triangle\, DEC$; therefore

(sector DEG : sector DEH) $> (\triangle\, DEF : \triangle\, DEC)$:

therefore (E. xxxiii. 6. and E. i. 6.)

$$(\angle DEF : \angle DEC) > (\overline{DF} : \overline{DC});$$

therefore (Supp. x. 5.)

$$(\angle FEC : \angle DEC) > (\overline{FC} : DC).$$

In the same manner it may be shewn that

$$(\angle AEC : \angle DEC) > (\overline{AC} : DC);$$

therefore (Supp. ix. 5.)

$$(\angle AED : \angle DEC) > (AD : DC);$$

that is, (E. xxxiii. 6.)

$$(\overset{\frown}{AB} : \overset{\frown}{BC}) > (AD : DC);$$

therefore (*constr.* and E. iii. 6.)

$$(\overset{\frown}{AB} : \overset{\frown}{BC}) > (\overline{AB} : \overline{BC}).$$

PROP. LXXXIV.

99. THEOREM. *The greater angle, at the base of a scalene triangle, has a greater ratio to the less angle, than the greater side has to the less side.*

Let ABC* be a scalene triangle, having the $\angle BCA$ greater than the $\angle BAC$:

Then $(\angle BCA : \angle BAC) > (\overline{AB} : \overline{BC})$.

About the $\triangle ABC$ describe (E. v. 4.) the circle $AECB$:

Then (E. xxxiii. 6.) $\angle BCA : \angle BAC :: \overset{\frown}{AB} : \overset{\frown}{BC}$:

But (Supp. lxxxiii. 6.) $(\overset{\frown}{AB} : \overset{\frown}{BC}) > (AB : BC)$;

therefore $(\angle BCA : \angle BAC) > (AB : BC)$.

PROP. LXXXV.

100. PROBLEM. *To divide a given circle into any required number of equal parts, by the circumferences of circles described within it, about its centre.*

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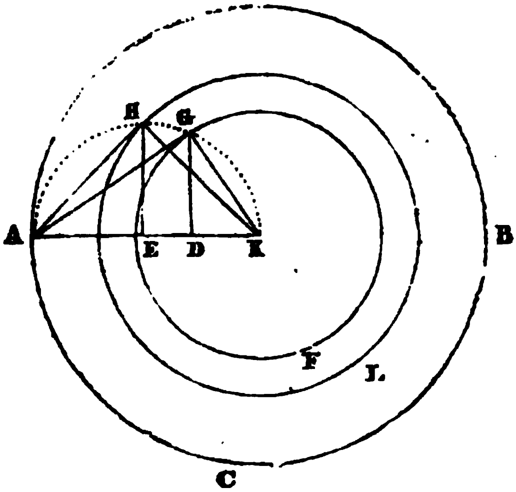
* See the Figure in p. 355.

Let *ABC* be a circle: It is required to divide



it into any number of equal parts, by the circumferences of circles described about the same centre with it.

Find (E. i. 3.) the centre *K* of the circle *ABC*; take any semi-diameter *KA*; divide (Supp. xlix. 1.) *KA* into as many equal parts, in the points *D*, *E*, &c. as those into which the given circle is to be divided; upon *AK*, as a diameter, describe the circle *AHK*; from *D*, *E*, &c. draw (E. xi. 1.) *DG*, *EH*, &c. perpendicular to *AK*, and meeting the circumference of *AHK* in *G*, *H*, &c.; join *K*, *G* and *K*, *H*, &c.; from the centre *K*, at the distances *KG*, *KH*, &c. describe the circles *GF*, *HL*, &c.; Then is the given circle divided in the required number of equal parts, by the circumferences of the circles so described.

For, join *A*, *H*, and *A*, *G*; and since (*constr.* and E. xxxi. 3.) the ∠s *AHK*, *AGK*, are right angles, and *HE*, *GD* are perpendicular to *AK*, therefore (E. viii. 6. and E. xx. 6. *Cor.* 2.)

$$\overline{AK} : \overline{KD} :: AK^2 : \overline{KG}^2.$$

But (E. ii. 12. and E. xxii. 6.)

$$AK^2 : KG^2 :: \text{circle } ABC : \text{circle } GF;$$

therefore (E. xi. 5.)

$$AK : KD :: \text{circle } ABC : \text{circle } GF.$$

Likewise,

$$\overline{KE} : AK :: \text{circle } HL : \text{circle } ABC;$$

therefore (E. xxiii. 5.)

$$KE : KD :: \text{circle } HL : \text{circle } GF;$$

therefore (E. xvii. 5.)

$$KD : \overline{DE} :: \text{circle } HL - \text{circle } GF : \text{circle } GF;$$

But (*constr.*) $KD = DE$; therefore the circle *GF* is equal to the space included between the circumferences of *GF* and *HL*: And, in the same manner, it may be shewn that this space is equal to the space included between the circumferences of *HL*, and of the circle next described according to the construction; and so on; therefore the circle *ABC* is thus divided into the required number of equal parts, by the circumferences of circles that have the same centre with it.

PROP. LXXXVI.

101. PROBLEM. *To find a circle, which shall be equal to the excess of the greater of two given circles above the less.*

Find (Supp. lxxv. 1.) a square which shall be equal to the excess of the square of the diameter of the greater circle above the square of the diameter of the less: It is manifest from E. ii. 12. and E. xvii. 5., that the side of the square so found will be the diameter of the circle, which is equal to the excess of the greater of the given circles above the less.

PROP. LXXXVII.

102. PROBLEM. *To find a circle to which a given circle shall have the same ratio, as that which one given straight line has to another.*

Find (E. xii. 6.) a fourth proportional (L) to the two given straight lines (A) and (B) and to the diameter (D) of the given circle; find, also, (E. xiii. 6.) a mean proportional (M) between the diameter (D) of the given circle, and the fourth proportional (L) first found;

therefore (E. xx. 6. *Cor.* 2.) $\overline{D}^2 : \overline{M}^2 :: D : L$;
and (*constr.*) $D : L :: A : B$;
therefore (E. xi. 5.) $D^2 : M^2 :: A : B$;

therefore (E. ii. 12.) the given circle has to a circle described on M, as a diameter, the same ratio as that which A has to B.

PROP. LXXXVIII.

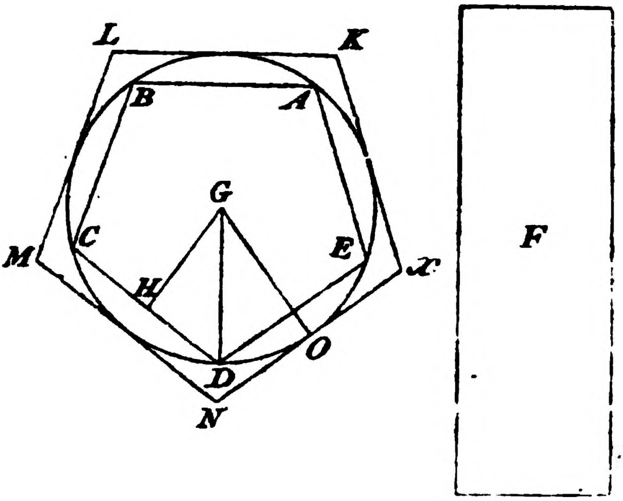
103. THEOREM. *If, in any given circle, two chords cut each other at right angles, the four circles described upon their segments, as diameters, shall, together, be equal to the given circle.*

For (Supp. l. 3.) the squares of the four segments are, together, equal to the square of the diameter: It is manifest, therefore, from E. xviii. 5. and E. ii. 12., that the circles described on the four segments of the chords are, together, equal to the given circle.

PROP. LXXXIX.

104. THEOREM. *A circle is equal to the half of the rectangle contained by its semi-diameter and by a straight line which is equal to its circumference.*

Let $ABCD$ be a circle, and let F be the half of



the rectangle contained by its semi-diameter and by a straight line equal to its circumference: The circle $ABCD$ is equal to the rectangle F.

For if it be not equal, it is either greater, or less, than it. If it be possible, let $F <$ the circle $ABCD$; therefore (E. ii. 12.) a polygon $ABCDE$ may be inscribed in the circle, which shall be greater than F.

Find (E. i. 3.) the centre G, and from G draw (E. xii. 1.) $\overline{GH}$ perpendicular to any side CD, of $ABCDE$, and join G, D. Then it may be assumed that the circumference of the circle is greater than the perimeter of the inscribed figure $ABCDE$; and (E. xvii. and xix. 1.) $\overline{GD} > GH$; therefore the rectangle contained by the circumference and the semi-diameter of the circle is greater than that contained by *GH*, and the perimeter of *ABCDE*, which latter rectangle (E. xli. 1. and E. i. 2.) is the double of the polygon *ABCDE*; therefore $F > ABCDE$; and it is also less; which is absurd.

But, if it be possible, let *F* be greater than the circle. Then (E. ii. 12.) a polygon *KLMNX* may be described about the circle, which shall be less than *F*; join the centre *G*, and any of the points of contact *O*; and since it may be assumed that the perimeter of *KLMNX* is greater than the circumference of the circle, the rectangle contained by the perimeter of *KLMNX* and $\overline{GO}$, which rectangle is the double of *KLMNX*, is greater than the rectangle contained by the circumference of the circle and *GO*; therefore the circumscribed polygon $KLMNX > F$; and it is also less; which is absurd. Therefore, the circle *ABCD* can neither be greater, nor less, than *F*; that is, it is equal to *F*.

105. Cor. The circumferences of circles are to one another as their semi-diameters.

Prop. XC.

106. Theorem. *A circle is a mean proportional between any regular polygon, described about it, and a similar polygon, the perimeter of which is equal to the circumference of the circle.*

For if there be taken a straight line (*P*) equal to the perimeter of the regular polygon described about the circle, and another straight line (*p*) equal to the perimeter of the similar polygon, or (*hyp.*) equal to the circumference of the circle, then (E. xx. 6. and

E. xxii. 6.) the polygon, described about the circle, is to the similar polygon, as P^2 is to p^2: But (Supp. ii. 4. *Cor.* 2.) the polygon, described about the circle, is the half of the rectangle contained by P and the circle's semi-diameter; and (Supp. lxxxix. 6.) the circle is the half of the rectangle contained by p; and by the circle's semi-diameter; therefore (E. i. 6.) that polygon is to the circle, as P is to p; and it has been shewn to be to the similar polygon, as P^2 is to p^2; therefore it has to the similar polygon a ratio, the duplicate of that which it has to the circle; therefore the circle is a mean proportional between the two similar polygons.

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A
SUPPLEMENT
TO THE
ELEMENTS OF EUCLID.

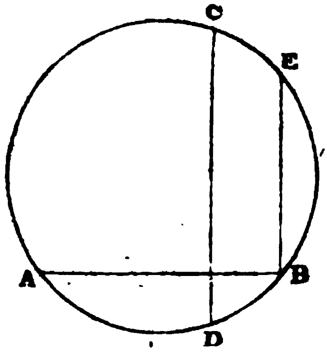
PART II.

BOOK I.

PROP. I.

1. **THEOREM.** *IF two chords of a circle cut each other at right angles, either pair of opposite arches, intercepted between them, is equal to the semi-circumference of the circle.*

In the circle ABC, let the chords AB, CD, cut



each other at right angles: $\overset{\frown}{AC}+\overset{\frown}{DB}$ shall be equal to the circumference of the circle.

For draw (E. xi. 1.) the chord *BE* perpendicular to *AB*; and therefore (E. xxviii. 1.) parallel to *DC*; wherefore (Supp. lv. 3.) $\overset{\frown}{ACE}$ is the semi-circumference; also (Supp. xxxv. 3.) $\overset{\frown}{DB}=\overset{\frown}{CE}$; therefore $\overset{\frown}{AC}+\overset{\frown}{DB}=\overset{\frown}{AC}+\overset{\frown}{CE}$, the semi-circumference.

2. SCHOLIUM. In like manner it may be shewn, that if two chords of a circle, produced, meet each other at right angles, the differences of the arches, intercepted between them, shall be equal to the semi-circumference of the circle. Thus it appears, that, if two straight lines, at right angles to each other, cut a circle, they shall cut off an arch equal to, less than, or greater than, the semi-circumference, accordingly as the angular point is in the circumference, is nearer to the centre than the circumference is, or is without the circle.

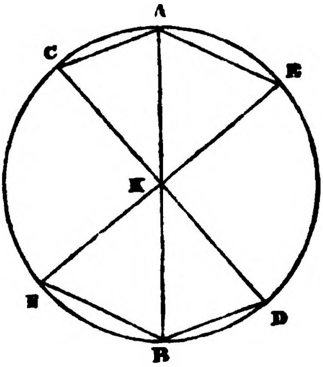
PROP. II.

3. THEOREM. *If three equal chords of a circle cut one another in the same point, within the circle, that point is the centre, and the chords are diameters of the circle.*

Let the three equal chords *AB*, *CD*, *EF*, of the circle *AFE* cut one another in the same point *K*, within the circle: *K* is the centre of the circle.

For draw $\overline{CA}$, $\overline{AE}$, $\overline{DB}$, $\overline{BF}$; then (hyp. and E. xxviii. 3.) $\overset{\frown}{BCA}=\overset{\frown}{DBC}$; take away the common part $\overset{\frown}{BFC}$, and $\overset{\frown}{AC}=\overset{\frown}{DB}$; therefore (E. xxix. 3.) $\overline{AC}=\overline{DB}$; also, (E. xxi. 3.) $\angle CAB=\angle CDB$, and

(E. xv. 1.) $\angle CKA = \angle BKD$; therefore (E. xxvi. 1.)



$KA = KD$. And, in the same manner, it may be shewn that $KF = KA$;

$$\therefore KA = KF = KD,$$

and (E. ix. 3.) K is the centre of the circle.

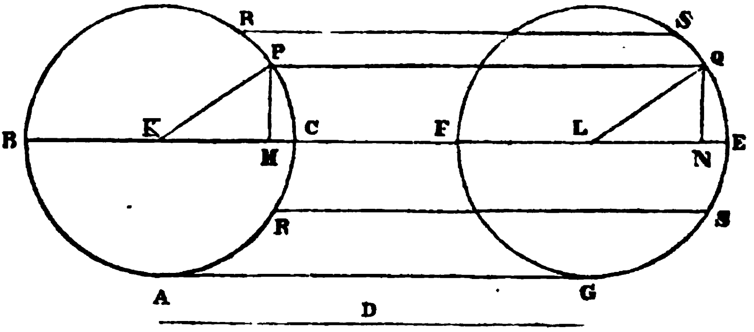
4. **Cor.** Through a point, within a given circle, which is not the centre, there cannot be drawn more than two equal chords.

Prop. III.

5. **Problem.** *To find the locus of the extremities of all the straight lines, that can be drawn from the circumference of a given circle, toward the same parts, each of them parallel and equal to a given finite straight line.*

ABC is a circle, *D* a finite straight line: it is required to find the locus of the extremities of all straight lines drawn from the circumference of the given circle towards the same parts, and equal and parallel to *D*.

Draw a diameter *BC* parallel to *D*, and in *BC* produced in the direction in which the straight lines are to be drawn, make $CE=D$: let K be the centre, from KE cut off $KL=D$. With L as centre, and at distance LE describe a circle EFG. The circle EFG is the locus required.



For, let a point P be taken any where in the circumference ABC; join P, K; draw $\overline{LQ}$ parallel to $\overline{PK}$, meeting $\overset{\frown}{EFG}$ in Q; join P, Q.

Since $KL=D$ and $CE=D$; $\therefore KL=CE$, and the part CL being common, $KC=LE$; but $KP=KC$, and $LQ=LE$; $\therefore LQ=KP$, and LQ was drawn parallel to KP, so that $\overline{PQ}$, KL join equal and parallel straight lines KP, LQ towards the same parts; therefore they are also equal and parallel; but KL is equal and parallel to D; therefore (E. xxxiii. 1.) also PQ is equal and parallel to D: that is, if from any point P in the circumference ABC, a straight line PQ be drawn equal and parallel to D, its extremity Q will be situated in the circumference EFG.

6. Cor. Since PK is parallel to QL, and KL meets them, therefore (E. xxix. 1.) $\angle PKC = \angle QLE$; therefore (E. xxvi. 3.) $\overset{\frown}{PC} = \overset{\frown}{QE}$: and, in like

manner it may be shewn, that, if any other straight line RS be drawn parallel to KL, $\overset{\frown}{RC}=\overset{\frown}{SE}$; wherefore, $\overset{\frown}{PR}$ and $\overset{\frown}{QS}$, which were the aggregates, or the differences, of equal arches, are equal to one another. Straight lines, therefore, drawn parallel to that which joins the centres of two equal circles, cut off, from their circumferences, equal arches.

PROP. IV.

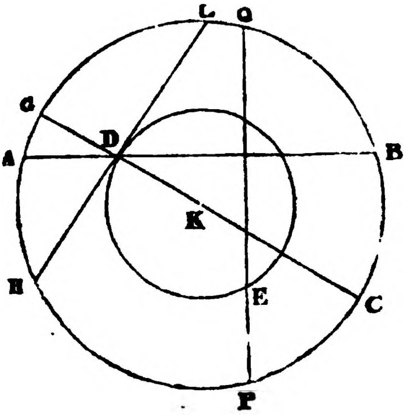
7. THEOREM. *If a chord of the greater of two concentric circles, cut the less of them, it's segments, between the two circumferences, shall be equal to one another.*

For if, from the common centre, a straight line be drawn at right angles to the chord, it shall (E. iii. 3.) bisect both the whole chord, and that part of it, which is a chord of the lesser circle. It is manifest, therefore, that, if from the halves of the greater of these two chords, there be taken the halves of the less, each from each, the remaining parts, which are the segments between the two circumferences, will be equal.

PROP. V.

8. THEOREM. *If any number of chords of the greater of two concentric circles, cut the less, the rectangles contained by the two segments, into which each of them is divided by the circumference of the less circle, shall be equal to one another.*

Let AB, PQ, be any two chords of the greater of two circles ABC, DEF, which have a common centre K; and let them be cut, by the circum-



ference of DEF, in the points D and E. Then $\overline{AD} \times \overline{DB} = \overline{PE} \times \overline{EQ}$.

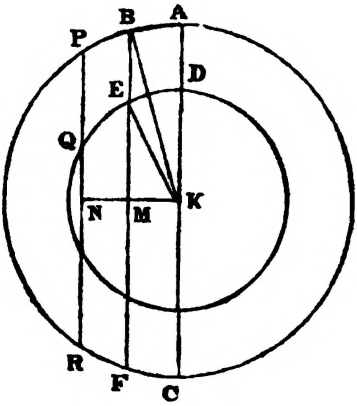
For join K, D, and produce KD, both ways, to meet the circumference of ABC in G and C. Wherefore (E. xxxv. 3.) $AD \times DB = \overline{GD} \times \overline{DC}$, which is the rectangle contained by the aggregate, and the difference, of the semi-diameters of the given circles; to which rectangle it may, in like manner, be shewn that $PE \times EQ$ is equal: therefore, $AD \times DB = PE \times EQ$.

9. Cor. If, through D, HL be drawn at right angles to KG, it shall (E. xvi. 3. *Cor.*) touch the circle DEF, and also (E. iii. 3.) shall be bisected in D. But (E. xxxv. 3.) $AD \times DB = \overline{HD} \times \overline{DL}$. Wherefore, $AD \times DB = DL^2$; and (E. xiv. 3.) DL is equal to any other tangent, drawn from any point in the circumference of DEF, to meet the circumference of ABC.

PROP. VI.

10. THEOREM. *Of parallel straight lines, terminated by the concave circumference of the greater of two concentric circles, and the convex circumference of the less, the least is that, which, produced, passes through the centre; and of the rest, that, which is nearer to the least, is always less than the more remote.*

Let AD, BE, PQ, be parallel straight lines,



terminated by the concave circumference of the circle ABE, and the convex circumference of DEF, which circles have a common centre K; and let AD, produced, pass through K: AD is the least, and BE, the nearer to AD, is less than PQ, the more remote. For, draw KMN, at right angles to BF and PR. Then (Supp. Part II. v. 1.)

$$\overline{AD}\times\overline{DC}=\overline{BE}\times\overline{EF}=\overline{PQ}\times\overline{QR}.$$

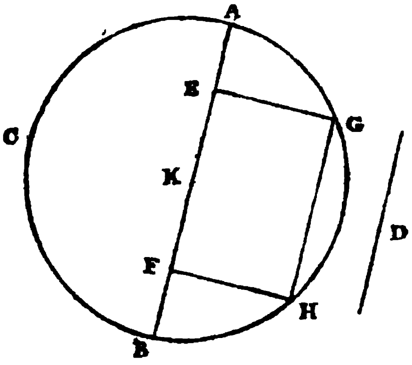
But (E. iii. 3. and xv. 3.) $KC>MF$, and $MF>NR$; also $DK>EM$, and $EM>QN$; therefore, $DC>EF$, and $EF>QR$; therefore $AD<BE$, and $BE<PQ$;

for if two *equal* rectangles be applied to each other, so that a right angle of the one may coincide with a right angle of the other, it is plain that if a side of the one be greater than a side of the other, the adjacent side of the former must be less than that of the latter.

PROP. VII.

11. PROBLEM. *To draw, in a given circle, a chord, which shall be equal and parallel to a given finite straight line, not exceeding the diameter of the circle.*

Let *ABC* be the given circle, and *D* a finite



straight line, not exceeding the diameter: it is required to draw in *ABC* a chord that shall be equal and parallel to *D*.

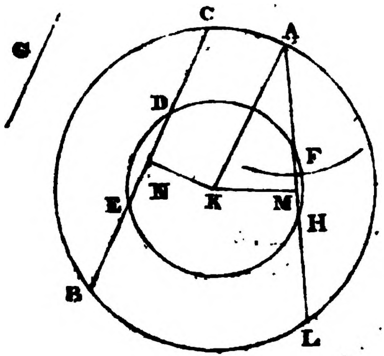
Find the centre *K* of *ABC*, and through *K*, draw the diameter *AB* parallel to *D*, if *AB*=*D*, the thing is done: if *AB*>*D*, then from *KA* and *KB* cut off *KE*, and *KF*, each equal to the half of *D*; from *E* and *F*, draw *EG* and *FH*, at right angles to *AB*, and let them meet the circumference in *G* and *H*. Then shall *GH* be equal and parallel to *D*.

For (E. iii. 3. and xiv. 3.) *EG* = *FH*; wherefore (E. xxxiii. 2.) *GH* is equal and parallel to *EF*; that is, (*constr.* and E. xxx. 1.) *GH* is equal and parallel to *D*.

PROP. VIII.

12. PROBLEM. *To place between the concave circumference of the greater of two given concentric circles, and the convex circumference of the less, a straight line which shall be equal and parallel to a given finite straight line, which is not greater than a tangent drawn from any point of the less to meet the greater circumference, and not less than half the difference of the diameters of the two circles.*

Let *ABC*, *DEF*, be the given circles, of which



K is the common centre, and let *G* be the given finite straight line: It is required to place between $\overset{\frown}{ABC}$ and $\overset{\frown}{DEF}$ a straight line equal and parallel to *G*.

Through *K* draw *KA* parallel to *G*, and meeting the outer circumference in *A*; from *A* as a centre, at a distance equal to *G*, describe a circle cutting $\overset{\frown}{DEF}$ in *F*; join *A*, *F*, and produce *AF* to meet $\overset{\frown}{DEF}$ and $\overset{\frown}{ABC}$, in *H* and *L*. From *K* draw *KM* at right angles to *AL*, and *KN* at right angles to *AK*, and make *KN* equal to *KM*. Lastly, through *N*, draw the chord *CNB*, at right angles to *KN*, and therefore (E. xxviii. 1. and E. xxx. 1.) parallel to *G*. Then shall *CD* be, also, equal to *G*.

For (E. iii. 3. and xiv. 3.) $CN = AM$, and $DN = FM$; therefore $CD = AF$; and (*constr.*) $AF = G$; therefore $CD = G$.

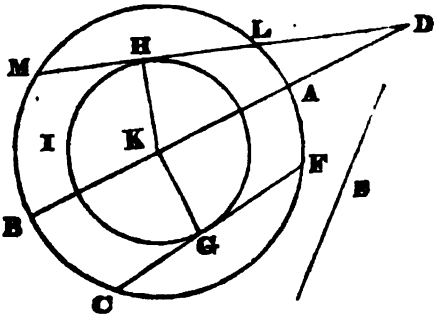
13. Cor. In like manner, may a straight line be placed between the concave circumferences of two given concentric circles, which shall be equal and parallel to a given finite straight line: but, in this case, the given finite straight line must not be greater than half the sum of the diameters of the two circles, and not less than a tangent drawn from any point of the less to meet the greater circumference.

Prop. IX.

14. Problem. *From a given point, without a given circle, to draw a straight line cutting the circle, so that it's segment, which is within the circle, shall be equal to a given straight line, not greater than a diameter of the circle.*

Let *ABC* be a circle, *D* a point without it, and *E* a straight line, not greater than a diameter of *ABC*: it is required to draw, from *D*, a straight line cutting the circle *ABC*, and having it's segment, within the circle, equal to *E*.

Find the centre *K*, and if *E* be equal to a diameter, it is plain, that the straight line joining *D*, *K*, and produced to meet the circumference again in *B*, is that, which was to be drawn. But, if *E* be less than a diameter, place (E. i. 4.) in the circle *ABC*, the chord *CF* equal to *E*. From *K* draw *KG* perpendicular to *CF*; from the centre *K*, at the distance *KG*, describe the circle *GHI*; and from *D* draw (E. xvii. 3.) *DLM* touching the circle *GHI* in *H*: Then shall *LM* be equal to *E*.



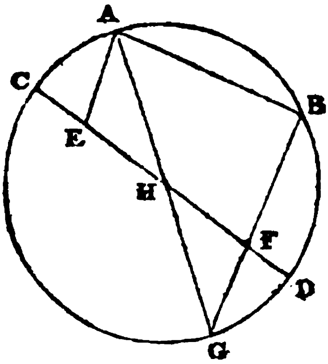
For (*constr.* and E. xviii. 3.) *KH* is at right angles to *LM*, and $KH = KG$; therefore (E. xiv. 3.)

$$LM = CF = E.$$

PROP. X.

15. THEOREM. *If from the extremities of any chord of a circle, straight lines be drawn at right angles to the chord, they shall meet any diameter of the circle in points, that are equidistant from the centre.*

Let *AB* be any chord, and *CD* any diameter, of the circle *ABC*, and let *AE*, *AF*, drawn at right angles to *AB*, meet *CD* in the points *E* and *F*:



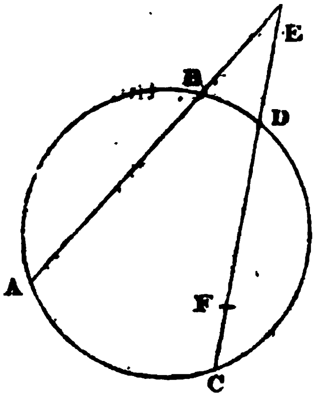
E and *F* are equidistant from the centre of the circle.

Produce *BF* to meet the circumference in *G*, join *A*, *G*, and let *AG* cut *CD* in *H*. And since (*hyp.*) the ∠*ABG* is a right angle, *AG* (Supp. lv. 3.) is a diameter, and therefore, *H* is the centre of the circle. Again (*hyp.* and E. xxviii. 1.) *AE* is parallel to *BG*; wherefore, (E. xxix. 1.) the ∠ *EAH* = ∠*HGF*, and (E. xv. 1.) ∠ *AHE* = ∠ *GHF*, and *HA* = *HG*; therefore (E. xxvi. 1.) *HE* = *HF*.

Prop. XI.

16. Theorem. *If two given finite straight lines, which are neither in the same straight line, nor parallel to each other, be produced so as to meet, and if the rectangle contained by the one line so produced, and the part of it produced, be equal to the rectangle contained by the other line produced, and the part of it produced, the extremities of the two given lines, shall be in the circumference of a circle.*

Let the two straight lines *AB*, *CD*, meet, when



produced, in *E*, and let $\overline{AE} \times \overline{EB} = \overline{CE} \times \overline{ED}$: the points *A*, *B*, *D*, *C*, shall be in the circumference of a circle.

Describe (E. v. 4.) a circle *ADB*, the circumference of which shall pass through *A*, *B* and *D*; it shall, also, pass through *C*. For it does not touch *EC* in *D*, because, then, (E. xxxvi. 3.) $AE \times EB$ would be equal to ED^2; but (*hyp.*) $AE \times EB = CE \times ED$, which exceeds ED^2: Wherefore, $\widehat{ABD}$ cuts *EC* again; and if it does not cut it in *C*, let it cut *EC* in any other point *F*; then (E. xxxvi. 3. *Cor.*) $\overline{AB} \times \overline{EB} = \overline{FE} \times \overline{ED}$; and (*hyp.*)

$$AB \times EB = CE \times ED; \therefore CE \times ED = FE \times ED,$$

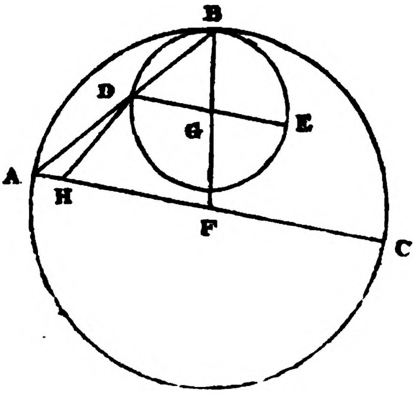
which is impossible. Wherefore, the circumference of *ADB* cannot cut *EC* again, in any other point than *C*.

PROP. XII.

17. THEOREM. *If, in any two given circles, that*

touch one another, there be drawn any two parallel diameters, an extremity of each diameter, and the point of contact, shall lie in the same straight line.

Let the circle *ABC*, the centre of which is *F*,



touch the circle *BDE*, the centre of which is *G*, in *B*, and let *AC*, *DE* be any two parallel diameters: the points *A*, *D*, *B*, are in the same straight line.

Join *B*, *D*, and *D*, *A*, and if *DA* be not in the same straight line with *BD*, let *BD*, produced, meet *AC* in *H*. Join *F*, *G*; then (E. xi. and xii. 3.) *FG* produced, passes through *B*. And (*hyp.* and E. xxix. 1.) $\angle BDG = \angle FHB$; but since *G* is the centre of *BDE*, $GB = GD$, and (E. v. 1.) $\angle BDG = \angle GBD$; wherefore, $\angle FBH = \angle FHB$, and (E. vi. 1.) $FH = FB = FA$; which is absurd; wherefore, *BD*, produced, cannot pass through any other point of *AC* than *A*; that is, *A*, *D*, *B*, are in the same straight line.

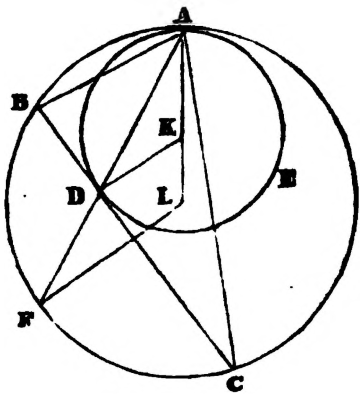
And the proof is the same, when the circles touch one another externally, only then the extremities of the diameters must be taken towards opposite parts.

18. Cor. Hence, if in any number of circles, that touch each other internally, in the same point, there be drawn diameters, all parallel to each other, the *locus*, of the corresponding extremities of the parallel diameters, will be a straight line passing through the common point of contact.

Prop. XIII.

19. Theorem. *If two circles touch each other internally, any chord of the greater circle, which touches the less, shall be divided, by the point of its contact, into segments that subtend equal angles at the point, in which the two circles touch one another.*

Let the two circles *ABC, ADE,* touch each other in *A*; also, let a chord *BDC,* of *ABC,* touch *ADE* in *D,* and let $\overline{AB}$, $\overline{AD}$, $\overline{AC}$, be drawn: the $\angle\, BAD = \angle\, CAD$. Find the centres *K* and *L* of the circles, and join *K, D,* and *L, K,* and produce *LK* (E. xi. 3.) to *A*; produce, also, *AD* to meet $\overset{\frown}{ABC}$ in *F,* and join *L, F.*



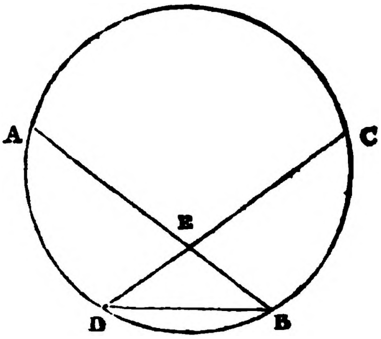
Then (*hyp.* and E. v. 1.) $\angle\, KAD = \angle\, KDA$, and $\angle\, LAF = \angle\, LFA$; $\therefore \angle\, KDA = \angle\, LFA$, and

(E. xxviii. 1.) *LF* is parallel to *KD*; but (*hyp.* and E. xviii. 3.) *KD* is at right angles to *BC*; wherefore (E. xxix. 1.) *LF* is at right angles to *BC*, and (E. xxx. 3.) bisects $\overparen{BFC}$; therefore, (E. xxvii. 3.) $\angle BAF = \angle CAF$.

Prop. XIV.

20. **Theorem.** *If two equal chords, in a given circle, cut one another, the segments of the one shall be equal to the segments of the other, each to each.*

In the circle *ABC*, let the two equal chords *AB*, *CD*, cut each other in *E*: $AE = EC$, and $DE = EB$.



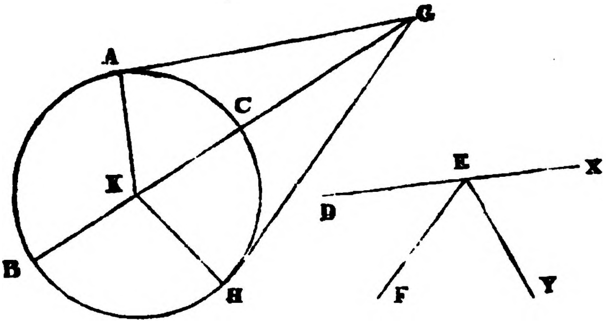
For join *D*, *B*: then (*hyp.* and E. xxviii. 3.) $\overparen{ADB} = \overparen{CDB}$; take away the common arch *DB*, and $\overparen{AD} = \overparen{CB}$; wherefore (E. xxvii. 3.) $\angle ABD = \angle CDB$; therefore (E. vi. 1.) $DE = EB$; and (*hyp.*) $AB = CD$; if, therefore, from these equals, there be taken *DE* and *EB*, it is plain that $AE = EC$.

Prop. XV.

21. **Problem.** *To draw two tangents to a given circle, which shall make with one another an angle,*

equal to a given angle; and shall meet in a given diameter produced.

Let *ABC* be the given circle, *K* being its centre



BC the given diameter, and *DEF* the given angle: it is required to draw two tangents to the circle *ABC*, which shall meet in *BC* produced, and which shall include an angle equal to *DEF*.

Produce *DE* to *X*; bisect the angle *FEX* by *EY*; at the point *K*, in *CK*, make (E. xxiii. 1.) the ∠s *CKA*, *CKH*, each equal to *FEY* or *XEY*; from *A* draw *AG* at right angles to *KA*, meeting *BC* produced in *G*; and join *G*, *H*; *GA* and *GH* are the tangents which were to be drawn.

For (*constr.* and E. iv. 1.) the ∠ *GHK* = ∠ *GAK*, which (*constr.*) is a right angle. Wherefore (E. xvi. 3. *Cor.*) *GA* and *GH* touch the circle.

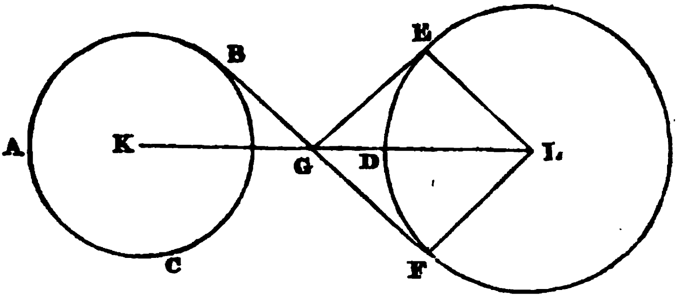
Again (E. xxxii. 1. *Cor.* 1.) the two ∠s *AKH*, *AGH*, are together equal to two right angles, as are, also, the two ∠s *DEF*, *FEK*, of which (*constr.*) the ∠ *FEK* = ∠ *AKH*: Wherefore the ∠ *AGH* = ∠ *DEF*.

PROP. XVI.

22. PROBLEM. *In the straight line, joining the centres of two given circles, that lie wholly without*

one another, to find a point, from which if two tangents be drawn to the two circles, they shall make equal angles with that line.

Let *KL* join the centres *K* and *L*, of the two circles *ABC, DEF*; it is required to find in *KL* a point, from which if two tangents be drawn, one to *ABC*, the other to *DEF*, they shall make equal angles with *KL*.



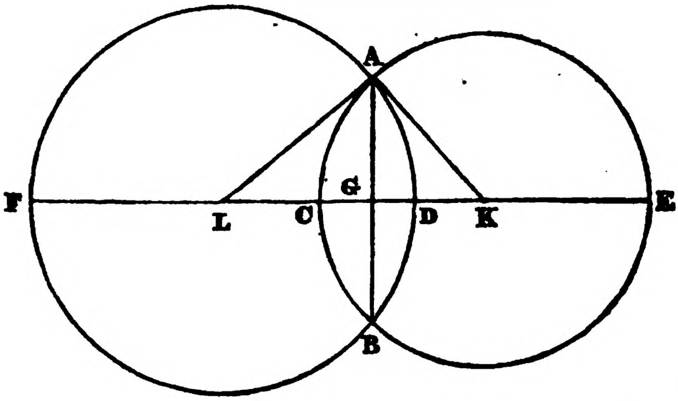
Draw (Supp. iii. 3.) *BF* touching the two given circles on contrary sides, and let it cut *KL* in *G*, and meet *DEF* in *F*: *G* is the point which was to be found.

For, from *G*, draw the tangent *GE*, and join *L*, *E*, and *L*, *F*. Then (E. xxxvi. 3.) $GE = GF$; also, $LE = LF$, and *GL* is common to the two △s *GEL, GFL*; therefore (E. viii. 1.) $\angle EGL = LGF$; and (E. xv. 1.) $\angle LGF = \angle BGK$; $\therefore \angle EGL = \angle BGK$.

Prop. XVII.

23. Theorem. *If two arches of unequal circles have a common chord, the arch, which belongs to the greater circle, shall have the less sagitta.*

Let the circles *ABE*, *ABF*, of which *ABF* is the



greater, cut each other in *A* and *B*, and let $\overset{\frown}{ADB}$, $\overset{\frown}{ACB}$, have a common chord *AB*: the sagitta of $\overset{\frown}{ADB}$ is less than the sagitta of $\overset{\frown}{ACB}$.

Find the centres, *K* and *L*, of *ABE* and *ABF*; join *L*, *K*; let *LK* cut *AB* in *G*, and produce *LK* to meet the circumferences in *E* and *F*: then (Supp. 3.) *LK* bisects *AB* at right angles, in *G*: Wherefore *GD* is the sagitta of $\overset{\frown}{ADB}$, and *GC* is the sagitta of $\overset{\frown}{ACB}$. Join *L*, *A* and *K*, *A*. And since (*hyp.*) $LA > KA$, $\overline{LA}^2 > \overline{KA}^2$; but (E. xlvii. 1.) $\overline{LG}^2$ is the excess of $\overline{LA}^2$ above $\overline{AG}^2$, and $\overline{KG}^2$ is the excess of $\overline{KA}^2$ above the same $\overline{AG}^2$;

$$\therefore \overline{LG}^2 > \overline{KG}^2; \quad \therefore LG > KG;$$

$$\text{also } FL > EK; \quad \therefore FG > EG.$$

But (E. xxxv. 3.)

$$\overline{FG} \times \overline{GD} = \overline{AG} \times \overline{GB} = EG \times GC;$$

$$\therefore \overline{FG} \times \overline{GD} = \overline{EG} \times \overline{GC};$$

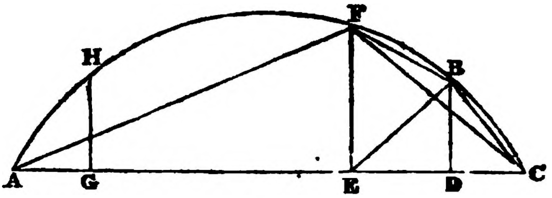
$$\text{and since } FG > EG, \quad GD < GC;$$

as was shewn in the course of the demonstration of Prop. 6.

PROP. XVIII.

24. THEOREM. *If the arch of a segment of a circle be divided into any two parts, the chord of the whole arch shall exceed the chord of the one part, by the double of the segment, which is cut off from the chord of the whole arch, by a straight line drawn at right angles to it, from the bisection of the other part.*

Let ABC be a segment of a circle, and let $\overset{\frown}{ABC}$,



the chord of which is $\overline{AC}$, be divided into any two parts in F; also let $\overset{\frown}{FC}$ be bisected in B, let BD be drawn at right angles to AC, and let A, F be joined,

$$\text{then } \overline{AC} = \overline{AF} + 2DC.$$

From DA cut off DE, equal to DC, and join E, B, and E, F, and F, B, and B, C; wherefore (*hyp*. and E. iv. 1.) $BE = \overline{BC}$, and (E. xxix. 3.)

$$\overline{BC} = \overline{BF}; \quad \therefore \overline{BF} = BE,$$

and (E. v. 1.) $\angle BFE = \angle BEF$;

but (E. xxii. 3.) $\angle AFB + \angle ACB$ is equal to two right angles, as is, also, $\angle AEB + \angle BEC$;

$$\therefore \angle AFB + \angle ACB = \angle AEB + \angle BEC,$$

and (*hyp*. and E. iv. 1.)

$$\angle BEC = \angle ECB \text{ or } \angle ACB;$$

$$\therefore \angle AFB = \angle AEB;$$

take from these the equal angles BFE, BEF, and there remains the $\angle AFE = \angle AEF$; therefore (E. vi. 1.) $AF = AE$;

$$\therefore AC = AE + EC = AF + 2DC.$$

25. Cor. In like manner, if $\overset{\frown}{AF}$ be bisected in H, and if HG be drawn at right angles to AC,

$$\overline{AC} = CF + 2AG;$$

$$\therefore 2\overline{AC} = \overline{AF} + \overline{FC} + 2(AG + DC),$$

$$\text{and } AC = \tfrac{1}{2}(AF + FC) + (AG + DC).$$

26. Scholium. The figure represents a segment less than a semi-circle; but the same proof is applicable to the other two cases, only if one of the points G, D, (suppose D,) should be in AC produced, then will AC be less than AF, instead of greater than it by twice AC. And, by the help of this Proposition, if any two arches and their chords, and the chord of their sum be given, the chord of the half of either of the arches may be found.

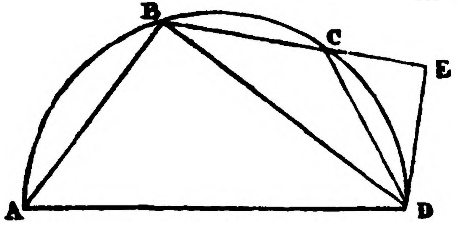
Prop. XIX.

27. Theorem. *If the curve of a semi-circle be subtended by any three chords, the square of the diameter shall exceed the squares of the three chords, by twice the rectangle contained by the middle chord, and the part of it produced, intercepted between its extremity, and the perpendicular drawn to it from the extremity of the diameter.*

Let the curve of the semi-circle $ABCD$ be subtended by the three chords AB, BC, CD; and from D, the end of the diameter AD, let DE be drawn at right angles to BC produced, and meeting it in E:

then, $\overline{AD}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + 2BC \times \overline{CE}$.

Join B, D; and, since (E. xxxi. 3.) the $\angle ABD$ is a right angle, (E. xlvii. 1.)



$$AD^2 = AB^2 + BD^2;$$

and (E. xii. 2.)

$$BD^2 = BC^2 + CD^2 + 2BC \times CE:$$

$$\therefore AD^2 = AB^2 + BC^2 + CD^2 + 2BC \times CE.$$

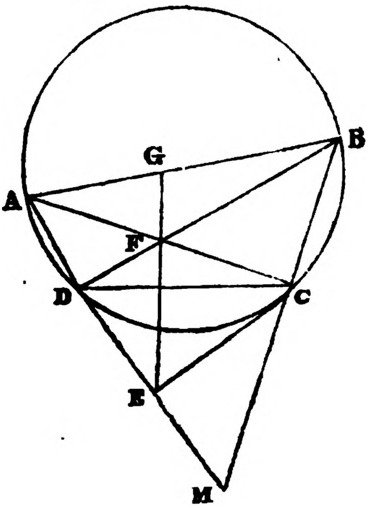
Prop. XX.

28. **Problem.** *If any number of quadrilateral rectilineal figures be inscribed in a given circle, having the same chord for their common base, and a diameter of the circle, for the side opposite to the base, to find the locus of the intersections of their two diagonals in each of them.*

Let $ABCD$ be the given circle, of which $\overline{DC}$ is a given chord: It is required to find the locus of the intersections of the diagonals in each of any number of quadrilateral rectilineal figures, inscribed in $ABCD$, having DC for their common base, and a diameter of the circle for the side opposite to it.

From D and C draw the two tangents DE, CE, and let them meet in E: then if a circle be described from E as a centre, at the distance ED, its circumference is the locus, which was to be found.

For let $ADCB$ be any such quadrilateral figure, having DC for its base, and any diameter AB, of the circle for its opposite side. Draw the diagonals AC, BD, cutting each other in F, and join E, F: produce DE to M, so that $DE = EM$, and join C, M.



The $\angle\, ADB$, (*hyp.* and E. xxxi. 3.) is a right angle; wherefore (E. xxxii. 1.) the $\angle\, DAB + \angle\, DBA$, is equal to a right angle; but (E. xxxii. 3.) the $\angle\, DAB = \angle\, FDE$, and (E. xxi. 3.) the $\angle\, DBA = \angle\, FCD$; therefore the $\angle\, FDE + \angle\, FCD$ is equal to a right angle. Again since $ED = EC = EM$, it is plain that a circle described upon DM as a diameter will pass through C; wherefore (E. xxxi. 3.) the $\angle\, DCM$ is a right angle;

$$\therefore \angle\, FDE + \angle\, FCD + \angle\, DCM,$$

is equal to two right angles; but $\angle\, FCD + \angle\, DCM$ is the $\angle\, FCM$ of the quadrilateral figure $DFCM$, the four angles of which are (E. xxxii. 1. *Cor.* 1.) together equal to four right angles; and $\angle\, FDM$

+ ∠ *FCM* has been shewn to be equal to two right angles; ∴ ∠ *DFC*+ ∠ *DMC* is equal to two right angles, and (*conv.*[*] E. xxii. 2.) a circle described about *DCM*, will also pass through *F*; but since *ED*=*EC*=*EM*, *E* is (E. xix. 3.) the centre of the circle so described; ∴ *EF* = *ED*. Therefore all the intersections of the diagonals are equidistant from *E*, and each of them will be found in the circumference of a circle described from *E* as a centre, at the distance *ED* or *EC*.

29. Cor. If *EF* be produced to meet *AB* in *G*, the angles at *G* are right angles. For since *EF*=*EC*, the ∠ *EFC*= ∠ *ECF* = (E. xxxii. 1.) ∠ *ABC*; but (E. xv. 1.) ∠ *AFG* = ∠ *EFC*; ∴ ∠ *AFG* = ∠ *ABC*, and ∠ *BAC* is common to the two triangles *AGF*, *ACB*; wherefore the third angle *AGF*, is equal to the third angle *ACB*, which (E. xxxi. 3.) is a right angle.

Prop. XXI.

30. Problem. *Through two given points in a diameter of a given circle, both on the same side of the centre, to draw two parallel chords, which shall intercept the two greatest arches.*

Let *F*, *G*, be two given points, in the diameter *AC* of the circle *ABCD*, both on the same side of the centre *E*: it is required to draw, through *F*, *G*, two parallel chords, which shall intercept the greatest arches.

Through *F*, *G*, draw the chords *HI*, *KL*, each at right angles to *AC*, and therefore (E. xxviii. 1.) parallel to each other: they shall intercept the greatest arches.

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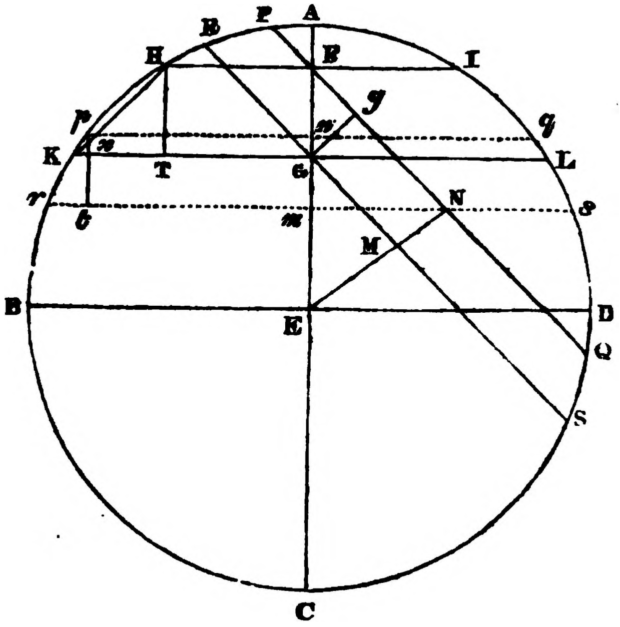
[*] The converse of E. xxii. 3. is easily demonstrated *ex absurdo*.



For through F, G, draw any other two parallel chords PQ, RS; from E draw EMN perpendicular to RS, and therefore also perpendicular to PQ. And since the ∠s EMG, ENF, are right angles,

$$EG > EM, \text{ and } EF > EN;$$

from EG, therefore, cut off Em equal to EM, and from EF cut off En equal to EN, and through m, n, draw the chords rms, pnq, parallel to HI or KL. Wherefore (*constr.* and E. xiv. 3.)

$$rs = RS, \text{ and } pq = PQ;$$

therefore (E. xxviii. 3.)

$$\overset{\frown}{rAs} = \overset{\frown}{RAS}; \text{ and } \overset{\frown}{pAq} = \overset{\frown}{PAQ}.$$

Therefore the difference of $\overset{\frown}{rAs}$ and $\overset{\frown}{pAq}$ is equal to the difference of $\overset{\frown}{RAS}$ and $\overset{\frown}{PAQ}$; and (Supp. xxxv. 3.)

the arches of a circle, included between two parallel chords, being equal to one another.

$$\overset{\frown}{pr} = \overset{\frown}{PR} = \overset{\frown}{QS} = \overset{\frown}{qs}.$$

Lastly, join H, K, and p, r; draw Gg at right angles to PQ; and from H, p, draw HT, pt, at right angles to KL and rs, respectively. Then, if pq fall between HI and KL, it is manifest that $\overline{HK}$ and pt will cut one another in some point X; wherefore (E. xvi. 1.) the $\angle\, Kxt$, (which is equal to the $\angle\, KHT$,) is greater than the $\angle\, rpt$; and, since

$$pt = nm = NM = Gg,$$

and that $HT = GF$, and $GF > Gg$;

therefore $HT > pt$; and it has been shewn that

$$\angle\, KHT > \angle\, rpt;\ \therefore\ HK > pr;$$

therefore (Supp. xlii. 3.) $\overset{\frown}{HK} > \overset{\frown}{pr}$ or $\overset{\frown}{PR}$.

But if pq fall between KL and rs, it is plain that

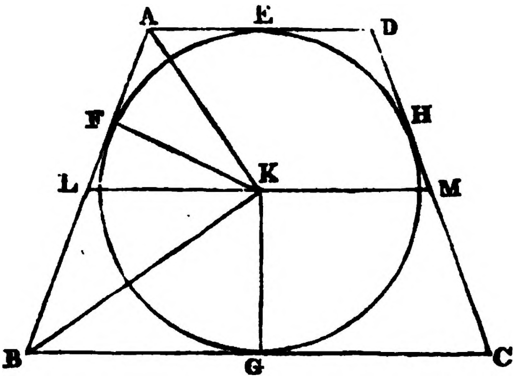
$$\overset{\frown}{pAs} > \overset{\frown}{HAL};\ \therefore\ \angle\, prt > \angle\, HKT;$$

$\therefore\ KHT > \angle\, rpt$ as before; and HT is also greater than pt; wherefore, as before, $\overset{\frown}{HK} > \overset{\frown}{pr}$, or $\overset{\frown}{PR}$.

PROP. XXII.

30. THEOREM. *If a quadrilateral rectilineal figure, having two of its sides parallel, be described about a circle, a straight line drawn through the centre of the circle, parallel to either of the two parallel sides, and terminated by the other two sides shall be equal to a fourth part of the perimeter of the figure.*

Let $ABCD$ be a quadrilateral rectilineal figure described about the circle $EFGH$, having the side AD parallel to BC; and through the centre K let there be drawn *LKM* parallel to *AD* or *BC*, and



meeting *AB* and *DC* in *L* and *M*: *LM* is equal to a fourth part of the perimeter of *ABCD*.

For let *AB* and *BC* touch the circle in *F* and *G*; join *K*, *A*, and *K*, *F*, and *K*, *B*, and *K*, *G*. Then (E. xxxvi. 3.) *BF*=*BG*, also *KF*=*KG*, and *BK* is common to the two △s *BFK*, *BGK*; therefore

(E. viii. 1.) ∠ *FBK*= ∠ *GBK*;

but (*constr*. and E. xxix. 1.) ∠ *GBK*= ∠ *BKL*;

∴ ∠ *LBK* = ∠ *LKB*,

and, (E. vi. 1.) *LK* = *LB*.

In the same manner it may be shewn, that *LK*=*LA*, that *KM*=*MC*=*MD*. Wherefore, *LM* is equal to the half of the aggregate of *AB* and *DC*, which aggregate is equal to the semi-perimeter of *ABCD*. Therefore, *LM* is equal to a fourth part of the perimeter of *ABCD*.

31. Cor. Of all quadrilateral rectilineal figures described about the same circle, and having two of their sides parallel, the square has the least perimeter.

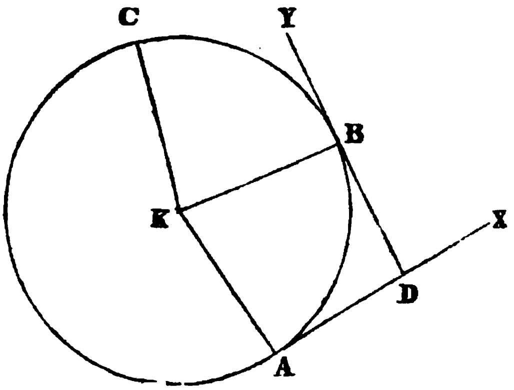
32. Def. Curve lines being supposed to be generated by the continual motion of a point, a curve

is said to be of *uniform curvature,* when its generating point, in describing equal arches, undergoes equal changes of direction.

Prop. XXIII.

33. Theorem. *The circumference of a circle is a curve of uniform curvature.*

The figure *ABC,* supposed to be described by the revolution of any finite straight line *KA,* about the fixt point *A,* in the same plane, is manifestly a plane figure contained by the line $\overset{\frown}{ABC}$, to which all straight lines, drawn from *K,* are equal to one another; therefore (E. Def. xv. 1.) it is a circle, having *K* for its centre, and $\overset{\frown}{ABC}$ for its circumference, which circumference is described by the motion of the point *A.*



The circumference *ABC* is a curve of uniform curvature.

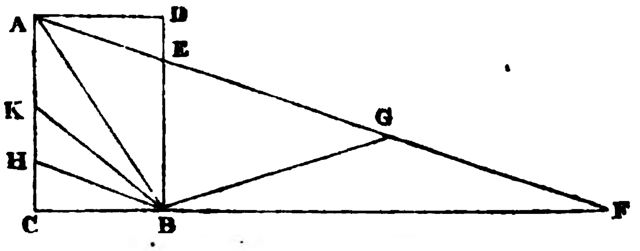
For let $\overset{\frown}{AB}$ be any arch of $\overset{\frown}{ABC}$; draw from the points *A* and *B,* the straight lines *AX, BY,* touching the circle, and join *K, A,* and *K, B.* Then

(E. xviii. 3.) the two ∠s *KAD, KBD,* are right angles; therefore (E. xxxii. 1. *Cor.* 1.) the two ∠s *AKB, BDA,* are together equal to two right angles, as (E. xiii. 1.) are also the two ∠s *BDA, BDX*; wherefore, the ∠ *BDX* = ∠ *AKB*: But the ∠ *BDX* measures the change of direction, which has taken place in the generating point between *A* and *B*. And if the generating point be supposed to describe another arch $\overset{\frown}{BC}$ equal to $\overset{\frown}{AB}$, and if *K, C* be joined, it may in like manner be shewn that the change in the direction of that point, is measured by the ∠ *CKB*. It is manifest, also, from the definition of the circle, that $\overset{\frown}{AB}$ may be applied to $\overset{\frown}{BC}$, so that they shall coincide; wherefore, the ∠ *CKB* = ∠ *BKA*: Therefore (32) the circumference is a curve of uniform curvature.

Prop. XXIV.

34. Theorem. *If, from any of the angles of any given rectangle, a straight line can be drawn cutting the two straight lines that contain the opposite angle, so that its segment, included between those two lines, shall be the double of the rectangle's diameter, then any given plane rectilineal angle can be trisected.*

For let *ABC* be any given plane rectilineal angle;



from any point *A*, in *AB*, draw *AC* a perpendicular to BC; and complete the rectangle CD: Then, if a straight line AEF, can be drawn from A, cutting DB, and CB produced in E and F, so that its segment EF shall be the double of AB, which is the diameter of CD, the $\angle ABC$ may be trisected.

At the point B, make (E. xxiii. 1.) the $\angle$s CBH, ABK, each equal to the $\angle AFC$; then shall the three $\angle$s ABK, KBH, HBC, be equal to one another.

For bisect EF in G, and join B, G; $\therefore BG$ (Supp. xxix. 1.) $= GE = GF$; therefore (E. v. 1, and E. xxxii. 1.) the $\angle AGB = 2 \angle AFB$; also the $\angle ABC = \angle BAF + \angle AFB$; but, since BG is equal to the half of EF, that is, (*hyp.*) to AB,

$$\text{the } \angle BAG, \text{ or } BAF = \angle AGB;$$

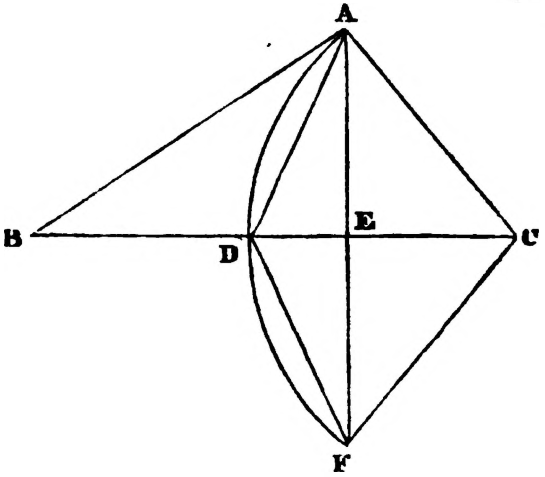
wherefore the $\angle ABC = \angle AGB + \angle AFC = 3 \angle AFC$: And the $\angle$s CBH, ABK, were made equal, each of them to the $\angle AFC$; therefore the $\angle KBH$ is also equal to the $\angle AFC$; so that the three $\angle$s ABK, KBH, HBC, into which the given $\angle ABC$ is divided, are equal to each other.

PROP. XXV.

35. THEOREM. *If from the hypotenuse of a right-angled triangle, a segment be cut off equal and adjacent to either of the other two sides, and if from the right angle a perpendicular be drawn to the hypotenuse, the straight line joining the right angle and the end of the segment, shall bisect the angle contained by the perpendicular and the third side of the triangle.*

From the hypotenuse BC, of the $\triangle ABC$, right-angled at *A*, let *CD* be cut off equal to *CA*; from *A*, let *AE* be drawn at right angles to *BC*, and let *A, D* be joined: *AD* bisects the $\angle BAE$.

Let a circle *ADF*, described from *C*, as a centre,



at the distance *CA*, or *CD*, cut *AE* produced in *F*, and join *D, F*, and *C, F*. And, since $CA = CF$, and *CE* is common to the two right-angled triangles *AEC, FEC*, (E. xlvii. 1.) $\overline{AE}^2 = \overline{FE}^2$; $\therefore AE = EF$; and *ED* is common to the two right-angled triangles *AED, FED*; therefore (E. iv. 1.) the $\angle DAE = \angle DFE$; but (*constr.* and E. xvii. 3. and E. xxxii. 3.)

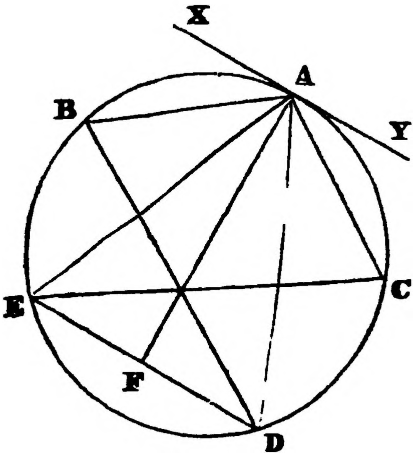
$$\angle BAD = \angle DFE; \quad \therefore \angle BAD = \angle DAE;$$

and *AD* bisects the angle *BAE*.

PROP. XXVI.

36. PROBLEM. *From a given point, in the circumference of a given circle, to draw a straight line, which shall touch the circle, by the help only of the* parallel ruler.

Let A be the given point, in the given circle



ABC; draw from A, any two chords AB, AC; from B draw BD parallel to AC, and from C draw CE parallel to AB; join D, E; and through A draw XAY parallel to DE: XAY shall touch the circle in A*.

From A draw AF perpendicular to ED, and join A, D, and A, E. Then (*constr.*) the ∠s ABD, ACE, being opposite angles of a parallelogram, are (E. xxxiv. 1.) equal to one another; and (E. xxi. 3.) $\angle ADE = \angle ACE$, and $\angle ABD = \angle AED$;

$$\therefore \angle ADE = \angle AED,$$

and the angles at F are right angles, and AF is common to the two △s AFD, AFE; $\therefore EF = FD$;

---

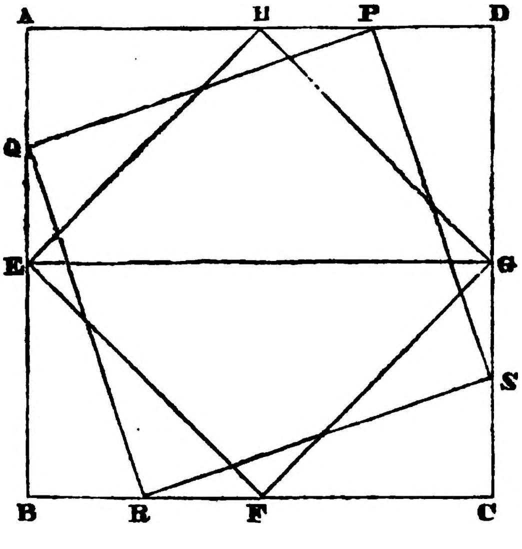
* The above method is, perhaps, the easiest *practical* method of drawing, with great exactness, a tangent to a circle, from a given point in the circumference: And it has the advantage of being a *general* method: For it is alike applicable to every one of the conic sections.

wherefore (E. iii. 3.) AF passes through the centre of the circle; and (*constr.*) XAY is parallel to ED, and therefore (E. xxix. 1.) at right angles to FA; wherefore (E. xvi. 3. *Cor.*) XAY touches the circle ABC, in the given point A.

PROP. XXVII.

37. THEOREM. *The least square, which can be inscribed in a given square, is that which is the half of the given square.*

Let $ABCD$ be the given square: If its four sides



be bisected in the several points E, F, G, H, and if these points be joined, $HEFG$ is (Supp. xxxv. 1.) a square: join E, G; then (*hyp.* and E. xxxiii. 1.) $AEGD$ is a parallelogram, and is the double (E. xli. 1.) of the $\triangle EGH$: and likewise, $EBCG$ is the double of the $\triangle EFG$; wherefore, $ABCD$ is the double of the square $EFGH$: let, now, $PQRS$ be any other square inscribed in $ABCD$; $PQRS > EFGH$.

For (E. xiii. 1.) the $\angle$s APQ, QPS, are equal to two right angles, as are also (E. xxxii. 1.) the $\angle$s DSP, PDS, DPS; and (*hyp.*) the $\angle QPS = \angle PDS$; $\therefore$ the $\angle APQ = \angle DSP$; and (*hyp.*) $QP = PS$; therefore (E. xxvi. 1.) the $\triangle$s PAQ, PDS, are equiangular, and $AQ = PD$. Wherefore, (E. xlvii. 1.)

$$\overline{PQ}^2 = \overline{AP}^2 + \overline{AQ}^2 = AP^2 + \overline{PD}^2;$$

but (E. ix. 2.)

$$AP^2 + PD^2 = 2\,\overline{AH}^2 + 2\,\overline{HP}^2:$$

and (E. xlvii. 1.)

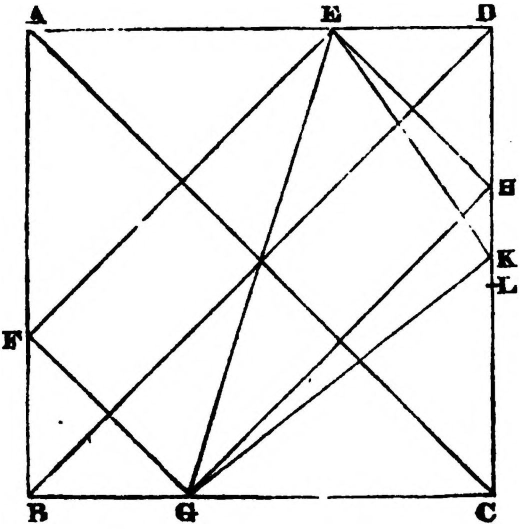
$$\overline{EH}^2 = AH^2 + \overline{AE}^2 = 2\,AH^2; \quad \therefore\ PQ^2 > EH^2.$$

PROP. XXVIII.

38. THEOREM. *If an oblong be inscribed in a square, each of its sides cuts off equal segments from the sides of the square.*

Let the oblong $EFGH$ be inscribed in the square $ABCD$; $AE = AF$, $ED = FB$, $CG = CH$, and $GB = HD$.

By reasoning as in Prop. 27, it may be shewn that $DH = FB$, and $ED = GB$; if, therefore, $ED = DH$, it is manifest that $AE = AF$, and $CG = CH$. But if ED be not equal to DH, one of them is the greater; let ED be greater than DH; from DC cut off DK, each equal to DE; join K, G, and E, G, and bisect DC, in L. And, since $DK = DE = GB$, and that $CD = CB$, $CK = CG$; and (E. xxxii. 1.) the $\angle$s DKE, CKG, at the bases of the isosceles right-angled △s *EDK*, *KCG*, are each of them half



of a right angle: Wherefore, (E. xiii. 1.) the ∠*EKG* is a right angle.

Therefore (E. xlvii. 1.)

$$\overline{EG}^2=\overline{EH}^2+\overline{HG}^2=\overline{EK}^2+\overline{KG}^2;$$

that is, (E. xlvii. 1.)

$$\overline{ED}^2+\overline{DH}^2+\overline{HC}^2+\overline{CG}^2=\overline{ED}^2+\overline{DK}^2+\overline{KC}^2+\overline{CG}^2;$$

from these equals take the common parts ED^2 and CG^2, and there remains $DH^2+HC^2=DK^2+KC^2$; which (E. ix. 2.) is impossible. Wherefore, *DE* is not unequal to *DH*; and therefore,

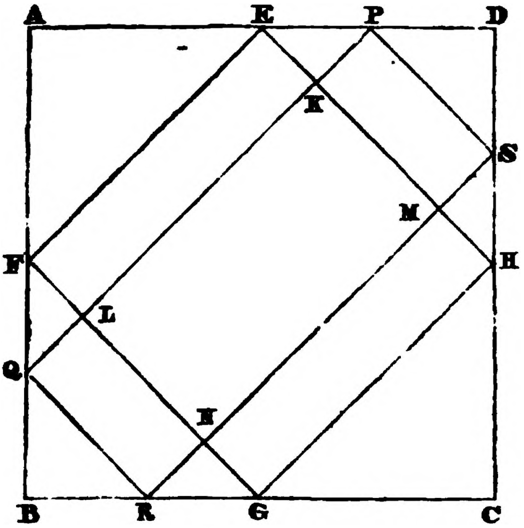
$$AE=AF,\ ED=EB,\ CG=CH,\ GB=HD.$$

39. Cor. If the diameters *AC*, *DB* be drawn, since the ∠*DEH*= ∠*DAC*, each of them being half of a right angle, *EH* (E. xxviii. 1.) is parallel to *AC*, and in the same manner, it may be shewn that *EF* is parallel to *DB*.

Prop. XXIX.

40. Theorem. *An oblong inscribed in a given square, is less than the least square inscribed in it.*

Let *PQRS* be any oblong inscribed in the square *ABCD*; and the sides of *ABCD* being bisected in *E, F, G, H,* and *EF, FG, GH, HE,* being drawn, *EFGH* is (Supp. Part II. xxvii. 1.) the least square that can be inscribed in *ABCD*; *PQRS* is less than *EFGH.*



Let *PQ* cut *EH* in *K*, and *FG* in *L*; and let *SR* cut *EH* in *M*, and *FG* in *N*. The ∠s *KEP, KPE* are (Supp. Part II. xxviii. 1. *Cor.* and *constr.*) each of them half of a right angle; therefore (E. vi. 1.) *EK*=*KP*; and *EF*, being equal to *EH*, is greater than *KM*; wherefore, the rectangle *FK* is greater than the rectangle *PM*; and, in the same manner, it may be shewn that MG is greater than QN; adding, therefore, KN to $FK+MG$, and also to $PM+QN$, it is manifest, that EG is greater than PR.

A SUPPLEMENT
TO THE
ELEMENTS OF EUCLID.

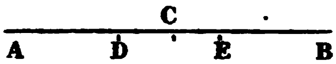
PART II.

BOOK II.

Prop. I.

1. **Problem.** *To divide a given finite straight line into two unequal parts, of which the less shall be a mean proportional between the greater, and the part which lies between the point of section, and the middle point of the given line.*

Let *AB* be a given line, and *C* its bisection: it is required, &c.



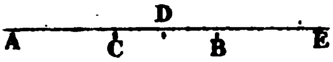
Trisect (Supp. xx. 1.) *AB* in the points *D* and *E*; then *BD* : *AD* :: *AD* : *DC*.

For (*constr.*) $BA : AC :: BD : AD$,
therefore (*permutando*) $BA : BD :: AC : AD$,
and (*dividendo*) $AD : BD :: DC : AD$;
therefore (*invertendo*) $BD : AD :: AD : DC$.

PROP. II.

2. PROBLEM. *A given straight line being divided into any two parts, to divide it again, so that the rectangle, contained by the whole line, and the part between the two points of section, shall be equal to the square of the part between the second point of section and the extremity, nearest to it, of the given line.*

Let the given straight line AB be divided in the point C: It is required to divide it again in D, so that $\overline{AB} \times \overline{CD} = \overline{DB}^2$.



To AB apply (E. xxix. 6.) a rectangle equal to $AB \times \overline{BC}$, and *exceeding* by $\overline{BE}^2$, and from BA cut off BD equal to BE: then $AB \times CD = DB^2$.

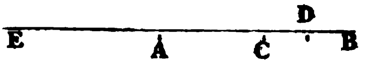
For (*constr.*) $AB \times BC = \overline{AE} \times \overline{EB}$; that is, (E. i. 2.) $AB \times \overline{BD} + AB \times CD = AB \times BE + BE^2$; and (*constr.*) $BE = BD$; $\therefore AB \times BD = AB \times BE$, and $\therefore AB \times CD = DB^2$.

PROP. III.

3. PROBLEM. *A given straight line being divided into any two parts, to divide it again, so that*

the rectangle, contained by the whole line, and the part between the points of section, shall be equal to the rectangle, contained by the two parts, into which the given line is divided by the second point of section.

Let the given straight line AB be divided in the



point C: It is required to divide it again in D, so that $\overline{AB} \times \overline{CD} = \overline{AD} \times \overline{DB}$.

Produce BA to E, and make AE equal to AB; to EB apply (E. xxviii. 6.) a rectangle equal to $AB \times \overline{BC}$, and *deficient* by $\overline{BD}^2$: then

$$AB \times CD = AD \times DB.$$

For (*constr.*) $AB \times BC = \overline{ED} \times DB$; that is,

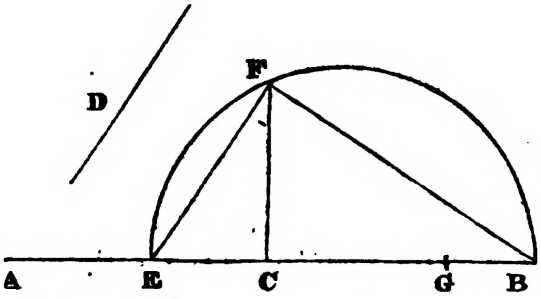
$$AB \times BD + AB \times CD = EA \times DB + AD \times DB =$$
$$\text{(because } EA = AB\text{)} \; AB \times BD + AD \times DB;$$
$$\therefore AB \times CD = AD \times DB.$$

Prop. IV.

4. **Problem.** *A straight line having been divided into two equal parts, to divide it again, so that the rectangle, contained by the two unequal parts, shall be to the square of the line between the two points of section, in a given ratio.*

Let AB be the given straight line, which has been bisected in C, and let the given ratio be that of AC, or BC, to D: It is required to divide AB again in G, so that $\overline{AG} \times \overline{GB} : \overline{CG}^2 :: AC : D$.

Find (E. xii. 6.) a fourth proportional to $(AC+D)$



D, and AC, and make CE equal to it; upon EB, as a diameter, describe the circle EFB; from C draw CF at right angles to EB, and let it meet $\overset{\frown}{EFB}$ in F; then (*hyp.* and E. vii. 3.) $CF < CB$; from CB, therefore, cut off CG equal to CF, and G is the point, which was to be found.

Join F, E, and F, B; and since (E. xxxi. 3.) the $\angle EFB$ is a right angle,

(E. viii. 6. *Cor.*) $BC : CF :: CF : CE$,

therefore (E. xix. 6. *Cor.*)

$$BC : CE :: \overline{BC}^2 : \overline{CF}^2 \text{ or } CG^2;$$

therefore (*constr.* and E. xi. 5.)

$$(BC+D) : D :: BC^2 : CG^2;$$

therefore (*dividendo*)

$$BC : D :: BC^2 - CG^2 : EG^2;$$

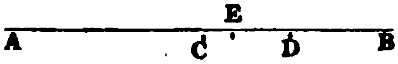
that is, (E. v. 2.) $BC : D :: AG \times GB : CG^2$.

PROP. V.

5. PROBLEM. *Two points* C, *and* D, *being given, in a finite straight line* AB, *which are neither of them one of its extremities, to find an intermediate*

point E, *such that the rectangle* $\overline{AE} \times \overline{EC}$, *shall be to the rectangle* $\overline{BE} \times \overline{ED}$, *in a given ratio.*

First, if the given ratio be a ratio of equality,



divide (E. x. 6.) CD, in E, so that

$$DE : EC :: AD : CB,$$

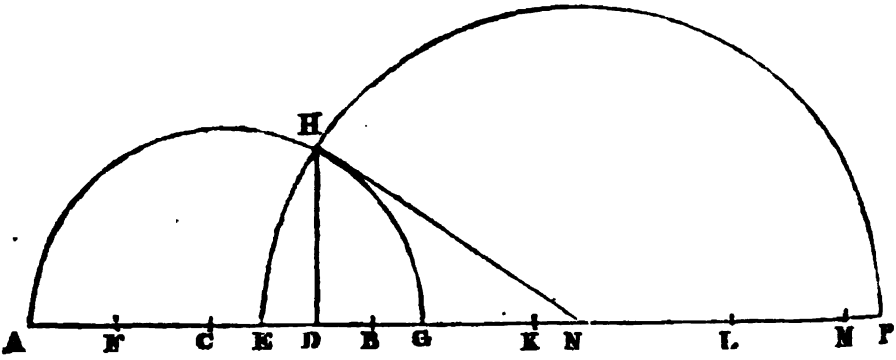
and E is the point, which was to be found.

For (*constr.* and E. xix. 5.)

$$AE : EB :: AD : CB :: DE : EC;$$

therefore (E. xvi. 6.) $AE \times EC = BE \times ED$.

But if the given ratio be not that of equality, let it be that of AC to CF; take (E. xii. 6.) DG a fourth proportional to AF, FC, and CD; upon AG, as a diameter, describe the circle AHG; draw DH at right angles to AG, and let it meet $\widehat{AHG}$ in H;



in AG produced take DK the double of DG; and in AK produced take KL a fourth proportional to AF, FC, and AC, and LM a fourth proportional to AF, AC and DB; lastly, bisect DM in N, join N,

H, and let the circumference EHP of the circle, described from N as a centre, at the distance NH, cut AB in E: E[*] is the point which was to be found.

For (*constr.* and E. xxxv. 3.)

$$\overline{AD} \times \overline{DG} = \overline{DH}^2 = \overline{PD} \times \overline{DE} = \overline{ME} \times \overline{ED};$$

therefore (E. xvi. 6.)

$$AD : EM :: ED : DG.$$

But (E. i. 6.)

$$AD : EM :: \overline{DA} \times \overline{AF} : ME \times AF;$$

$$\therefore ED : DG :: DA \times AF : ME \times AF;$$

but (*constr. permutando* and *invertendo*),

$$CD : AF :: DG : FC;$$

$$\text{and } CD : AF :: \overline{CD} \times AD : AF \times AD;$$

$$\therefore DG : FC :: AD \times \overline{DC} : DA \times AF;$$

and, (as hath been shewn,)

$$ED : DG :: DA \times AF : ME \times AF;$$

therefore (*ex æquali*,)

$$ED : FC :: AD \times DC : ME \times AF.$$

But (E. i. 2.) $ME \times AF =$

$$DE \times AF + \overline{DK} \times AF + \overline{KL} \times AF + \overline{LM} \times AF;$$

and (*constr.* and E. xvi. 6.)

$$DK \times AF = 2\,\overline{FC} \times CD, \text{ and } \overline{KL} \times \overline{AF} = \overline{AC} \times \overline{CF},$$

$$\text{and } LM \times AF = AC \times \overline{DB};$$

$$\therefore ED : FC :: AD \times DC : (DE \times AF + 2\,FC \times CD + AC \times CF + AC \times DB).$$

Again, $DE : FC :: DE^2 : ED \times FC$;

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[*] It is easily shewn, *ex absurdo*, that the point E, as determined by the construction, falls between the two given points C and D.

therefore $AD \times DC$ is to the last mentioned five rectangles, as $DE^2 : ED \times FC$, and (*componendo*),

$$DE : FC :: AD \times DC + DE^2 : (ED \times FC + ED \times AF + 2FC \times CD + AC \times CF + AC \times DB);$$

but $DE \times FC + DE \times AF = DE \times CA$,
and $AC \times DB + AC \times ED = AC \times EB$;

$$\therefore DE : FC :: AD \times DC + DE^2 : (AC \times EB + 2FC \times CD + AC \times CF).$$

Also (E. i. 6.)

$$DE : FC :: (AC + 2CD) \times DE : (AC + 2CD) \times CF :: AC \times DE + 2CD \times DE : AC \times CF + 2DC \times CF;$$

$$\therefore AD \times DC + DE^2 : AC \times EB + 2FC \times CD + AC \times CF :: AC \times ED + 2CD \times DE : AC \times CF + 2FC \times CD;$$

therefore (E. xix. 5.) DE is to FC as the excess of $(AD \times DC + ED^2)$ above $(AC \times CD + 2CD \times DE)$ is to $AC \times EB$; that is, (E. vii. 2.) as $AE \times EC$ is $AC \times EB$: But (E. i. 5.)

$$AC : ED :: AC \times EB : BE \times ED;$$

therefore (*ex æquo*),

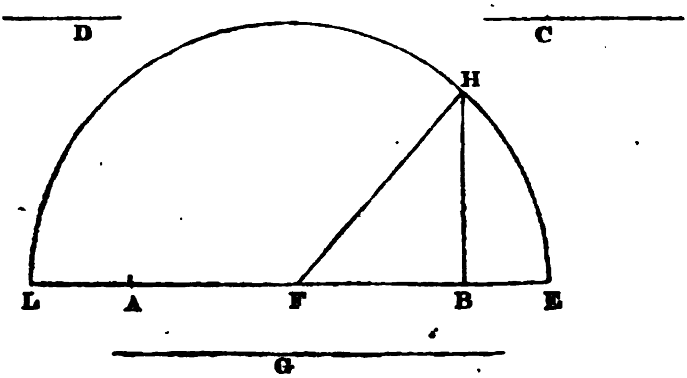
$$AC : CF :: AE \times EC : BE \times ED.$$

PROP. VI.

6. PROBLEM. *To produce a given straight line, so that the square of the line, which is made up of the whole and the part produced, together with the square of the part produced, shall be in a given ratio to a given square.*

Let AB be the given straight line, and C the side of the given square; and let the given ratio be that of AB to D: It is required to produce AB to E, so that $(\overline{AE}^2+\overline{EB}^2) : C^2 :: AB : D$.

Bisect AB in F; find (Supp. xxi. 6.) a square which shall be to C^2, as BF is to D, and let G be the side of it; from B draw BH at right angles to AB, and make BH equal to the side of a square, which is the excess of G^2 above the half of $\overline{AB}^2$ (E. xiv. 2. and Supp. lxxv. 1.); join F, H, and from the centre F, at the distance FH, describe the circle EHL, cutting AB produced in E and L; then is E the point to which AB was to be produced.



For (E. xxxv. 3.)

$$\overline{LB}\times\overline{BE}=\overline{BH}^2, \text{ that is, } AE\times EB=BH^2;$$

$$\therefore AE\times EB+\tfrac{1}{2}AB^2=BH^2+\tfrac{1}{2}AB^2=G^2;$$

but (*constr.*) $BF : D :: G^2 : C^2$;

$$BF : D :: (AE\times EB+\tfrac{1}{2}AB^2) : C^2;$$

and (E. iv. 5.)

$$2BF \text{ or } AB : D :: (2AE\times EB+\overline{AB})^2 : C^2;$$

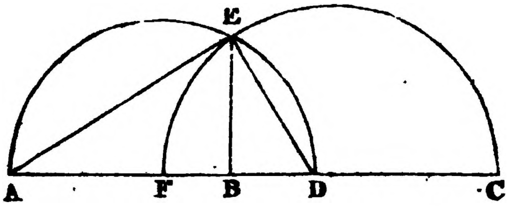
but (E. vii. 2.) $AB^2+2AE\times EB=AE^2+EB^2$;

$$\therefore (AE^2+EB^2) : C^2 :: AB : D.$$

PROP. VII.

7. **PROBLEM.** *To produce a given straight line, so that the square of the given line, together with twice the rectangle contained by the given line and the part produced, shall be to the square of the part produced, in a given ratio.*

Let AB be the given straight line, and let the given ratio be that of AB to BD, BD being placed in the same straight line with AB: It is required to produce AB to C, so that



$$(\overline{AB}^2 + 2AB \times \overline{BC}) : \overline{BC}^2 :: AB : BD.$$

Upon AD, as a diameter, describe the circle AED; from B draw BE at right angles to AD, meeting $\widehat{AED}$ in E; join D, E, and from the centre D, at the distance DE, describe the circle FEC, the circumference of which cuts AB produced in C: Then is C the point to which AB was to be produced.

Join A, E; and (E. xxxi. 3.) the $\angle AED$ is a right angle; also (E. xxxv. 3.)

$$\overline{BE}^2 = AB \times \overline{BD} = \overline{FB} \times BC;$$

therefore (E. xvi. 6.)

$$AB : BC :: FB : BD;$$

therefore (*componendo*),

$$AC : BC :: FD, \text{ or } DE : BD,$$

and (E. xxii. 6.)

$$\overline{AC}^2 : BC^2 :: \overline{DE}^2 : BD^2;$$

but (E. viii. 6. and E. xx. 6.)

$$DE^2 : BD^2 :: AD : BD;$$

therefore (E. xi. 5.)

$$AC^2 : BC^2 :: AD : BD;$$

that is, (E. iv. 2.)

$$(AB^2 + 2AB \times BC + BC^2) : BC^2 :: AD : BD;$$

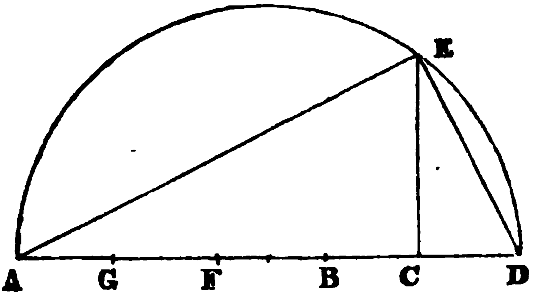
therefore (*dividendo*),

$$(AB^2 + 2AB \times BC) : BC^2 :: AB : BD.$$

PROP. VIII.

8. **PROBLEM.** *To divide a given finite straight line into two parts, so that the rectangle, contained by the whole line, and the difference of those two parts, shall be in a given ratio to the square of the less of the two parts.*

Let AB be the given straight line, and let the given ratio be that of R to S: It is required to divide it into two parts in F, so that



$$\overline{AB} \times (AF - BF) : \overline{FB}^2 :: R : S.$$

Produce AB to C, so that BC is equal to a fourth proportional to R, S, and AB; again, produce AC to D, so that $CD=BC$; upon AD, as a diameter, describe the circle AED; from C, draw CE at right angles to AD, meeting $\overset{\frown}{AED}$ in E; and from CA cut off CF equal to CE: F is the point of division, which was to be found.

Join A, E, and D, E; and from FA cut off FG equal to FB; then, (E. xxxi. 3. and E. viii. 6.)

$AC : CE :: CE : CD,$
that is, (*constr.*) $AC : CF :: CF : CB;$
therefore (*dividendo*) $AF : CF :: FB : BC,$
and (*permutando*) $AF : FB :: CF : BC;$
therefore (*dividendo*) $AG : GF :: FB : BC;$
therefore (E. xvii. 6.) $\overline{AG}\times\overline{BC}=\overline{GF}\times FB=FB^2;$
therefore (E. vii. 5.)

$$\overline{BA}\times AG : FB^2 :: BA\times AG : AG\times BC$$
$$:: BA : BC, \text{ (E. i. 6.).}$$

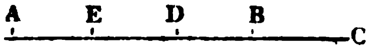
But $AG=(AF-FB)$, and (*constr.*)

$$BA : BC :: R : S;$$
$$\therefore AB\times(AF-FB) : FB^2 :: R : S.$$

PROP. IX.

9. PROBLEM. *To divide a given finite straight line into two parts, the less of which shall be in a given ratio to their difference.*

Let AB be the given straight line, and AB to



BC the given ratio: It is required to divide AB in D, so that $DB : (AD-DB) :: AB : BC.$

Take AD equal to a fourth proportional to $(AB+AC)$, AC, and AB; then is D the point of division which was to be found.

For from DA cut off DE equal to DB; then (*hyp.*)

$$AB+AC : AC :: AB : AD,$$

therefore (*dividendo*) $AB : AC :: DB : DA$,
and (*permutando*) $AB : DB :: AC : DA$;
again (*dividendo*) $AD : DB$ or $DE :: CD : DA$,
and (*dividendo*) $AE : ED :: CD-DA : DA$;
but (*hyp.* and *permutando*)

$$AB+AC : AB :: AC : AD;$$

therefore (*dividendo*) $AC : AB :: CD : DA$;
and (*dividendo*) $BC : AB :: CD-DA : DA$;
therefore (E. xi. 5.) $AE : ED :: BC : AB$;
and (*invertendo*) ED or $DB : AE :: AB : BC$,
that is, (*constr.*) $DB : (AD-DB) :: AB : BC$.

10. SCHOLIUM. If the given ratio be a ratio of equality, it is manifest that the straight line AB is, by this construction, cut in extreme and mean *arithmetic* ratio; which is more easily effected, by trisecting the given line.

PROP. X.

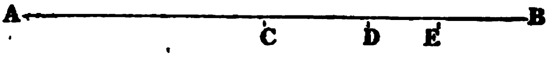
11. PROBLEM. *To divide a given finite straight line into three parts, that shall be continual proportionals, and the square of the greatest of which shall be equal to the squares of the other two.*

Upon any assumed finite straight line, as an hypotenuse, describe (Supp. lxxiii. 6.) a right-angled triangle, having its three sides continual proportionals, and place its three in one straight line: If then the given straight line be divided (E. x. 6.) similarly to that line, it is manifest, that that which is required, will have been done.

PROP. XI.

12. PROBLEM. *To divide a given finite straight line into four parts, which shall be proportionals, according to a given ratio, and shall, also, be such, that, if the least be added to each of the three least, the greatest, and the three aggregates, shall be proportionals.*

Let AB be the given straight line, and let the



given ratio be that of R to S: It is required, &c.

Bisect AB in C; make CD a fourth proportional to R, S and BC; and divide BD in E, so that

$$BE : ED :: R : S;$$

then are AC, CD, BE, ED, the parts into which AB was to be divided.

For (*constr.* and E. xi. 5.)

$$CB \text{ or } AC : CD :: BE : ED :: R : S.$$

Again (*constr.*) $BE : ED :: R : S$;

therefore (*componendo*) $BD : ED :: R+S : S$,

and $BD : 2ED :: R+S : 2S$.

Also (*constr.*) $CB : CD :: R : S$;

therefore (*convertendo*) $CB : DB :: R : R-S$,

and (*constr.* and *componendo*)

$$DB : BE :: R+S : R,$$

therefore (*ex æquo*) $CB : BE :: R+S : R-S$;

therefore (*convertendo*) $CB : CE :: R+S : 2S$.

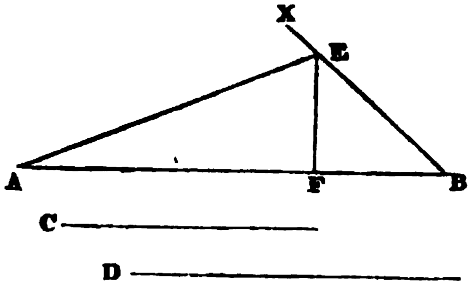
$$\text{Wherefore } CB \text{ or } AC : CE :: BD : 2ED,$$
$$\text{and } CE = CD + DE,\ BD = BE + DE,$$
$$2ED = ED + ED.$$

13. **Cor.** From the demonstration it is manifest, that if the three least of any four proportionals be together equal to the greatest, the greatest, and the aggregates of each of the others and the least, shall also be proportionals.

Prop. XII.

14. **Problem.** *To divide a given finite straight line into two parts, such that the aggregate of their squares shall be in a given ratio to a given square.*

Let AB be the given finite straight line, C the side



of the given square, and let the given ratio be that of R to S: It is required to divide AB in F, so that

$$(\overline{AF}^2 + \overline{FB}^2) : C^2 :: R : S.$$

From B draw $\overline{BX}$, making with AB an angle equal to the half a right angle; find (Supp. xxi. 6.) the side D of a square, such that

$$D^2 : C^2 :: R : S;$$

from A, as a centre, at a distance equal to D, describe a circle, which, if the problem be possible, will cut BX; let it cut BX in E, and from E draw EF perpendicular to AB: F is the point which was to be found.

For (*constr.*) the $\angle EFB$ is a right angle, and the $\angle EBF$ the half of a right angle; therefore (E. xxxii. 1.) the $\angle FEB = \angle EBF$, and (E. vi. 1.) $EF = FB$; also, (*constr.* and E. xlvii. 1.)

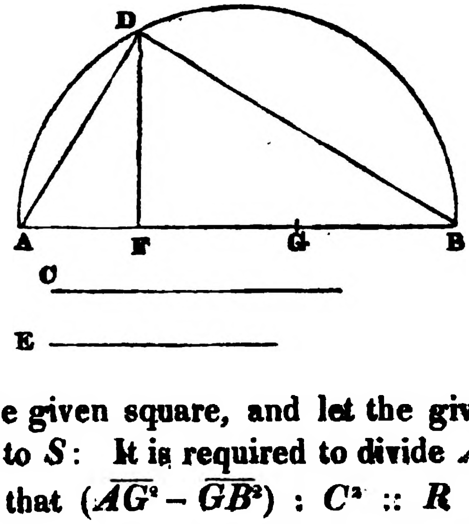
$$\overline{AE}^2 = D^2 = AF^2 + \overline{FE}^2 = AF + FB^2;$$

therefore (*constr.*) $AF^2 + FB^2 : C^2 :: R : S$.

PROP. XIII.

15. PROBLEM. *To divide a given finite straight line into two parts, such that the difference of their squares shall be in a given ratio to a given square.*

Let AB be the given finite straight line, C the side of the given square, and let the given ratio be that of R to S: It is required to divide AB in G,



so that $(\overline{AG}^2 - \overline{GB}^2) : C^2 :: R : S$.

Upon AB as a diameter describe the circle ADB; find (Supp. xxi. 6.) the side E of a square, such that

$$E^2 : C^2 :: R : S;$$

in the circle ADB inscribe $\overline{AD}$ equal to E; from D, draw DF perpendicular to AB, and bisect FB in G: G is the point which was to be found.

Join D, B; then (E. iv. 2.)

$$AG^2=\overline{AF}^2+\overline{FG}^2+2AF\times FG;$$

but $2AF\times FG=AF\times\overline{FB}=\overline{DF}^2$ (E. viii. 6.)

$$\therefore AG^2=AF^2+\overline{FD}^2+FG^2,$$

that is, (E. xlvii. 1.)

$$AG^2=AD^2+FG^2=AD^2+GB^2;$$

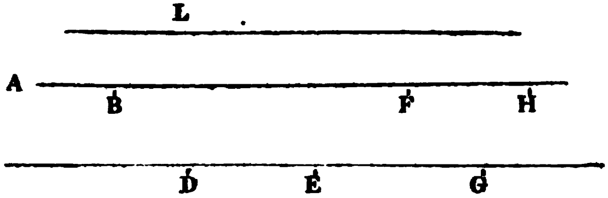
$$\therefore AG^2-GB^2=AD^2:$$

Wherefore, (*constr.*) $AG^2-GB^2 : C^2 :: R : S$.

PROP. XIV.

16. PROBLEM. *To divide a given finite straight line into four proportionals, so that the aggregate of the extremes shall be in a given ratio to the aggregate of the means.*

Let L be the given straight line to be divided, and let the given ratio be that of AD to AB: It is required, &c.



In CD produced, take any point E, and in AB, produced, make AF equal a fourth proportional to AB, CD and CE: Also, in CE, produced, make EG equal to a fourth proportional to DE, AB and BF; and, again, in AB produced, BH equal to a fourth proportional to AB, CE and EG: if then the line L be divided into four parts, (E. x. 6.) proportional to AB, CE, EG and BH, it will be divided as is required.

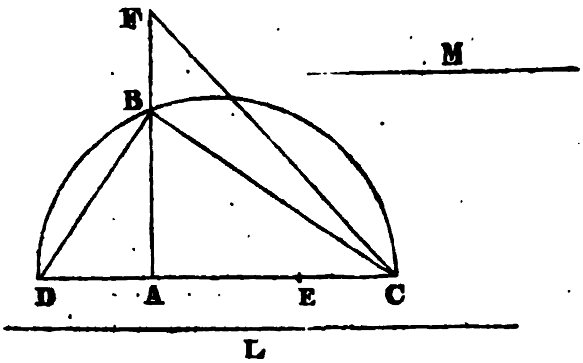
For (*constr.*) $DE : BF :: AB : EG$,
and $AB : CE :: EG : BH$;
therefore (*permutando*) $AB : EG :: CE : BH$;
therefore (E. xi. 5.) $DE : BF :: CE : BH$,
and (*invertendo*) $BH : CE :: BF : ED$.
But (*constr.*) $BF : ED :: EG : AB$;
therefore (E. xi. 5.) $BH : CE :: EG : AB$.
Again, $BH : BF :: CE : DE$;
therefore (*dividendo*) $BH : FH :: CE : CD$,
and (*permutando*) $BH : CE :: FH : CD$;
therefore $FH : CD :: EG : AB$,
and $FH : EG :: CD : AB$.
But (*constr.*) $AB : CE :: CD : AF$,
therefore (*permutando* and *invertendo*)
$CD : AB :: AF : CE$;
therefore (E. xi. 5.) $AF : CD :: FH : EG$,
and (E. xii. 5.) $AH : EG :: CD : AB$;

and (*constr.*) AB, CE, EG, BH are proportionals; and AH, the aggregate of the extremes, to EG the aggregate of the means in the given ratio. If, therefore, L be divided into four parts, proportional to AB, CE, EG and BH, they will also be proportional, and the aggregate of the extremes will be to the aggregate of the means, in the given ratio.

PROP. XV.

17. PROBLEM. *To divide a given finite straight line into four proportionals, so that the difference of the extremes shall be to the difference of the means, in a given ratio.*

Let L be the given finite straight line, and let the ratio be that of M to AB: it is required, &c.



From A draw any straight line AC, at right angles to AB; join B, C, and from B draw BD at right angles to BC, meeting CA produced in D; about the right-angled △ DBC describe the circle CBD; from AC, cut off AE a fourth proportional to $(M+AC)$, AC, $(M+AD)$; and produce AB to F, so that AF shall be equal to a fourth proportional to AB, AC, AE: if, then, L be divided into four parts proportional to AB, AC, AE and AF, it will be divided as is required.

For (*constr*. and E. viii. 6. and E. xvii. 6.)

$$\overline{DA}\times\overline{AC}=\overline{AB}^2;$$

and (*constr.*) $M+AC : AC :: M+AD : AE$;

$$\therefore M\times\overline{AE}+AE\times AC=M\times AC+AC\times\overline{AD}.$$

Again (*constr.*) $AB : AC :: AE : AF$;

$$\therefore \overline{FA}\times AB=\overline{CA}\times AE;$$

$$\therefore FA\times AB+AE\times\overline{M+AC}$$

$$=CA\times AE+AC\times\overline{M+AD};$$

$$\therefore FA\times AB+EA\times M=AC\times M+AC\times AD.$$

But (E. i. 2.) $AB\times AF=AB^2+AB\times\overline{BF}$,

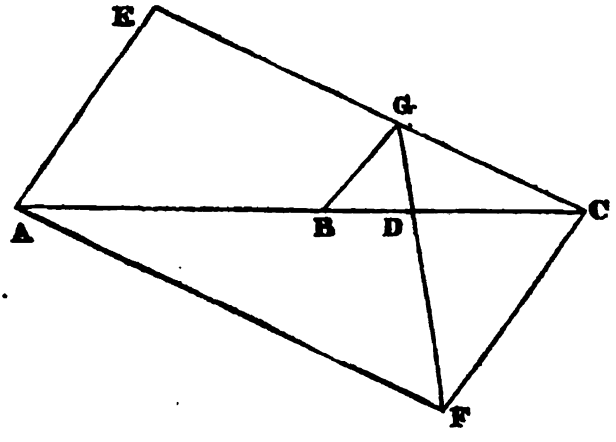
$$\text{and } AB^2 = DA \times AC;$$
$$\therefore AC \times M = AE \times M + AB \times BF.$$
$$\text{Also } AC \times M = M \times AE + M \times EC;$$
$$\therefore M \times EC = AB \times BF;$$

therefore (E. xvi. 6.) $FB : EC :: M : AB$; and FB is the difference of the extremes AB, AF, and EC is the difference of the means AC and AE; wherefore it is manifest that if L be divided into four parts, proportional to AB, AC, AE, and AF, these parts will also be proportionals, and the difference of the extremes will be to the difference of the means, in the given ratio.

Prop. XVI.

18. Problem. *Three points, in a given straight line being given, to find, in that line, a fourth point, between the second and third, such that the rectangle contained by the distance between the first and fourth, and the distance between the second and fourth, shall be equal to the square of the distance between the third and the fourth points.*

Let A be the first, B the second, and C the third



of three given points, all in the same straight line ABC: It is required to find in it a fourth point D, such that $\overline{AD} \times \overline{DB} = \overline{DC}^2$.

Make (E. xxxi. 1.) AC the diameter of any parallelogram EF; through B draw BG parallel to AE, or EF meeting EC in G; join G, F, and let GF cut AC in D: D is the point which was to be found.

For (*constr.* E. xv. 1. E. xxix. 1.) the △s ADF, GDC, are equiangular, as are, also, the △s GDB, CDF;

therefore (E. iv. 6.) $AD : DC :: FD : DG$,
and $FD : DG :: DC : DB$;
therefore (E. xi. 5.) $AD : DC :: DC : DB$;
therefore (E. xvii. 6.) $AD \times DB = DC^2$.

Prop. XVII.

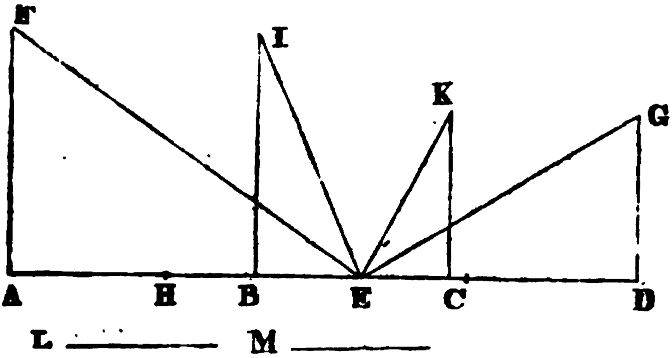
19. **Problem.** *Four points being given in the same straight line, to find, in that line, a fifth point between the second and third, such that the square of the distance between the first and the second, together with the square of the distance between the first and the fifth, shall be to the square of the distance between the third and the fourth, together with the square of the distance between the fourth and the fifth, in a given ratio.*

Let A, B, C, D, be four given points in the straight line AD: It is required to find in it a fifth point E, such that $(\overline{AB}^2 + \overline{AE}^2)$ shall be to $(\overline{CD}^2 + \overline{DE}^2)$ in the given ratio of L to M.

From A and D, draw AF and DG, at right angles to AD, and make AF equal to AB, and DG equal to DC: divide (Supp. xxii. 6.) AB in H, so that

$$\overline{AH}^2 : \overline{HB}^2 :: L : M;$$

and (Supp. xxxviii. 6.) find a point E in AD, such that $\overline{FE} : \overline{GE} :: AH : HB$;



E is the point which was to be found.

For (*constr.*) $FE : GE :: AH : HB$;
therefore (E. xxii. 6.) $FE^2 : GE^2 :: AH^2 : HB^2$,
that is, (*constr.* and E. xii. 5.)
$FE^2 : GE^2 :: L : M$;
that is, (E. xlvii. 1. and *constr.*)
$(AB^2 + AE^2) : (DC^2 + DE^2) :: L : M$.

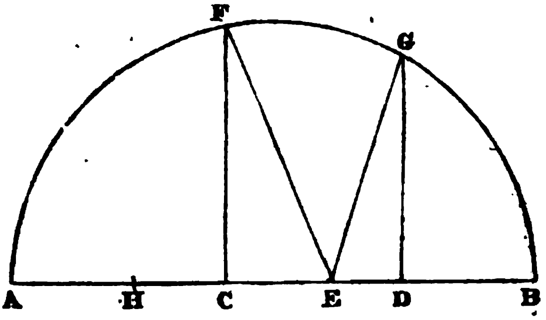
20. Cor. If BI and CK be drawn at right angles to AD, and if BI be made equal to AB, and CK to DC, it is manfest that, in like manner, a point E may be found, such that $(\overline{AB}^2 + \overline{BE}^2)$ shall be to $\overline{DC}^2 + \overline{CE}^2$, in a given ratio.

Prop. XVIII.

21. Problem. *Two points being given, in a given finite straight line, to find a third intermediate point, such that the rectangle contained by the two parts into which the line is divided by the first given point, together with the square of the distance between that point and the third point, shall be in a given ratio to*

the rectangle contained by the parts, into which the line is divided by the second given point, together with the square of the distance between that point and the third point.

Let the two points *C, D* be given, in the finite straight line *AB*: It is required to find a third point *E*, between *C* and *D*, such that $(\overline{AC} \times \overline{CB} + \overline{CE}^2)$ shall be to $(\overline{AD} \times \overline{DB} + \overline{DE}^2)$ in the given ratio of *L* to *M*.



Upon *AB* as a diameter, describe the circle *AFGB*; divide (Supp. xxii. 6.) *AC* in *H*, so that

$$\overline{AH}^2 : \overline{HC}^2 :: L : M;$$

from *C* and *D*, draw *CF, DG* at right angles to *AB* meeting $\overset{\frown}{AFGB}$ in *F* and *G*, respectively; and (Supp. xxxviii. 6.) find in *CD*, a point *E*, such that

$$FE : GE :: AH : HC;$$

E is the point which was to be found.

For it may be shewn, as in the preceding proposition, that

$$\overline{FE}^2 : \overline{GE}^2 :: L : M;$$

that is, (E. xlvii. 1.)

$$\overline{FC}^2 + \overline{CE}^2 : \overline{GD}^2 + \overline{DE}^2 : L : M;$$

but (E. iii. 3. and E. xxxv. 3.)

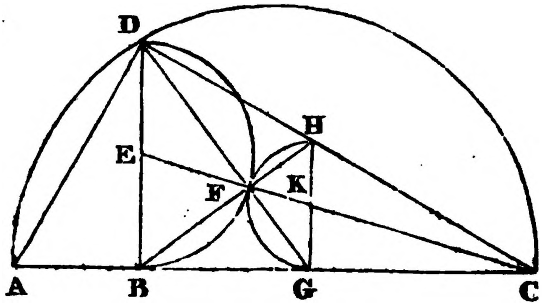
$$\overline{FC}^2 = \overline{AC} \times \overline{CB}, \text{ and } \overline{GD}^2 = \overline{AD} \times \overline{DB};$$

$$\therefore (\overline{AC} \times \overline{CB} + \overline{CE}^2) : (\overline{AD} \times \overline{DB} + \overline{DE}^2) : L :: M.$$

PROP. XIX.

22. **PROBLEM.** *The first of three continual proportionals being given, and also the aggregate of the other two, to find those other two.*

Let AB be the first of these proportionals, and



let BC, placed in the same straight line with AB, be the aggregate of the second and third of them: It is required to find the second and the third.

Upon AC, as a diameter, describe the circle ADC; from B draw BD at right angles to AC, meeting $\overset{\frown}{ADC}$ in D; bisect DB in E; and upon DB, as a diameter, describe the circle DFB; join E, C, and let EC cut $\overset{\frown}{DFB}$ in F; also join D, F, and B, F, and let DF produced, meet BC in G: then are BG, GC, the second and third of the proportionals which were to be found.

Draw $\overline{AD}$, $\overline{DC}$, $\overline{BF}$; also draw GH at right angles to BC, meeting DC in H, and EC in K;

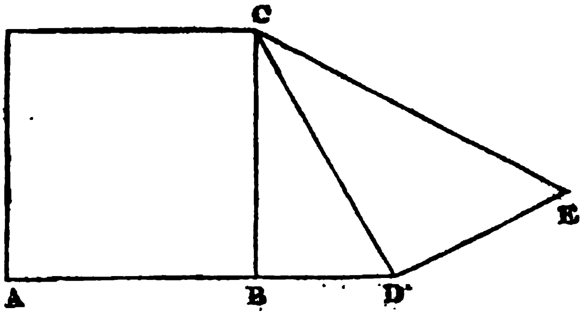
therefore (E. xxviii. 1.) *GH* is parallel to *BD*, and (*constr.* and Supp. lxi. 1.) is bisected in *K*: join *F*, *H*; and since (E. xv. 1.) ∠*KFG* = ∠*DFE* = (E. v. 1.) ∠*EDF* = (E. xxix. 1.) ∠*FGH*; ∴ *KF* = *KG*, and the circumference of a circle, described from *K*, as a centre, at the distance *KB*, will pass through *G* and *H*, and will touch the circle *DFB* in *F*;

therefore (Supp. xix. 6.) *DB* : *BG* :: *BG* : *GH*;
also (E. iv. 6.) *AB* : *BD* :: *HG* : *GC*;
therefore (*ex æquo*) *AB* : *BG* :: *BG* : *GC*.

PROP. XX.

23. PROBLEM. *To find two squares, the difference between which shall be in a given ratio to the difference between the less of them, and a given square.*

Let *AC* be the given square, and let the given ratio be that of *AB* to *BD*, *BD* being placed in the same straight line with *AB*: It is required to find two squares, the difference between which shall be to the difference between the less of them and *AC*, as *AB* is to *BD*.



Join C, D; from D draw DE, at right angles to CD, and make it equal to a mean proportional between AB and BD, and join C, E: $\overline{CE}^2$ and $\overline{CD}^2$ are the squares which were to be found.

For (*constr.* and E. i. 6.) $\overline{DE}^2 = \overline{AB} \times \overline{BD}$; therefore (E. i. 6.) $DE^2 : BD^2 :: AB : BD$; but (E. xlvii. 1.)

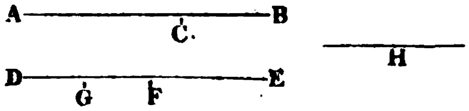
$$DE^2 = CE^2 - CD^2, \text{ and } BD^2 = CD^2 - CB^2;$$

$$\therefore CE^2 - CD^2 : CD^2 - CB^2 :: AB : BD.$$

PROP. XXI.

24. THEOREM. *Two unequal magnitudes being given, an intermediate magnitude may be found, which shall be commensurable with a third given magnitude, of the same kind.*

Let AB, AC, be two given unequal magnitudes,



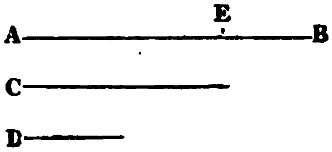
DE a third given magnitude, of the same kind with them: A magnitude may be found greater than AC and less than AB, which shall be commensurable with DE.

Bisect DE in F, DF in G, and so on, until (E. i. 10. *Barrow.*) a part DG shall have been found less than CB; and let H be the least multiple of DG, which exceeds AC; then is H less than AB: for, if not, a part equal to DG might be taken from H, and there would remain a multiple of DG greater than AC, because DG is less than CB; which is contrary to the supposition. Wherefore *H* is less than *AB* and greater than *AC*, and *DG* measures both *H* and *DE*.

Prop. XXII.

25. **Theorem.** *Two incommensurable magnitudes being given, a third magnitude may be found, commensurable with one of them, and the difference between which and the other, shall be less than any assigned magnitude, the magnitudes being all of the same kind.*

Let *AB* and *C* be two given incommensurable magnitudes, and let *D* be any assigned magnitude, of the same kind with them: Another magnitude may be found commensurable with *C*, and the difference between which and *AB* shall be less than *D*.



For let *AE* be the difference between *AB* and *D*; then (Prop. 19.) a magnitude may be found greater than *AE* and less than *AB*, which shall be commensurable with *C*; and it is plain, that the difference between this magnitude and *AB*, is less than *D*.

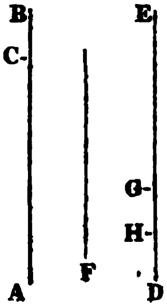
26. **Scholium.** Thus it appears, that, although two magnitudes be incommensurable, a third magnitude may be found, approaching indefinitely near to a state of equality with either of them, which

shall be commensurable with the other: and there is not, perhaps, in the whole compass of mathematics, a more remarkable truth than this. Cavallerius says of it, *Hæc sunt Geometriæ admiranda, quæ, etsi in infiniti abditis recessibus lateant, ejusdem tamen vim et efficaciam non effugiunt.*

PROP. XXIII.

27. PROBLEM. *Two unequal finite straight lines being given, to find another intermediate straight line which shall be incommensurable with a third given finite straight line.*

Let AB, AC, DE, be the three given finite straight lines: It is required to find a straight line F, greater than AC and less than AB, which shall be incommensurable with DE.



From ED cut off EG equal to the side of a square, of which ED is the diameter; in like manner, from GD cut off GH equal to the side of a square, of which GD is the diameter, and so on; and since (E. xix. 1.) $EG > \frac{1}{2} ED$, $GH > \frac{1}{2} GD$, and so on, a remainder HD (E. i. 10.) will, at length, be found, which is less than CB: Let F be the least multiple of HD, which is greater than AC; then, as in Prop. 19, it may be shewn that F is less than AB.

Also, (*constr.*) $ED : GD :: GD : HD$;

$ED : HD :: \overline{ED}^2 : \overline{GD}^2$;

but (*constr.*) ED and EG are incommensurable; wherefore (E. xvii. 13.) ED and GD are incommensurable, and also (E. ix. 10.) ED^2 and GD^2 are incommensurable; wherefore (E. x. 10.) ED and HD are incommensurable, and F, which (*hyp.*) is a multiple of HD, is also incommensurable with ED; and it has been shewn to be greater than AC and less than AB.

28. Cor. Hence, if two commensurable finite straight lines be given, a third straight line may be found, which shall be incommensurable with either of them, and the difference between which and the other, shall be less than any assigned straight line.

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SUPPLEMENT

TO THE

ELEMENTS OF EUCLID.

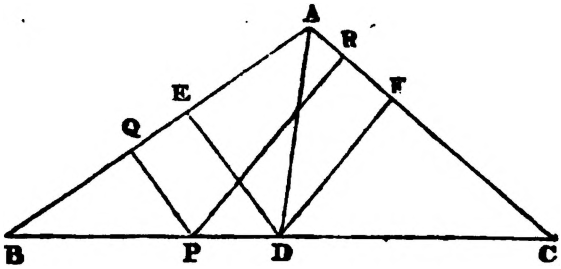
PART II.

BOOK III.

PROP. I.

1. THEOREM. *The two sides of a triangle are to one another, reciprocally, as the perpendiculars drawn to them from the bisection of the base.*

From the bisection, *D*, of the base, *BC*, of the △*ABC*, let there be drawn *DE* at right angles to *AB*, and *DF* at right angles to *AC*: Then



$$AB : AC :: DF : DE.$$

Join A, D. Then (E. xxxviii. 1.) the $\triangle ADB = \triangle ADC$; also (E. xli. 1.) $\overline{AB} \times \overline{ED}$ is the double of the $\triangle ADB$, and $\overline{AC} \times \overline{FD}$ is the double of the $\triangle ADC$;

$$\therefore \overline{AB} \times \overline{DE} = \overline{AC} \times \overline{DF};$$

therefore (E. xvi. 6.) $AB : AC :: DF : DE$.

2. **Cor.** If P be any other point in the base BC, of the $\triangle ABC$, and if PQ be drawn perpendicular to AB, and PR perpendicular to AC, it may, in like manner, be shewn that

$$AB : AC :: \overline{BP} \times \overline{PR} : \overline{CP} \times \overline{PQ}.$$

Prop. II.

3. **Theorem.** *If perpendiculars be drawn from the three angles of an acute-angled triangle, to the opposite sides, the rectangles contained by the several sides and their corresponding segments, shall be equal: Also, the rectangle contained by the two segments of any one of the perpendiculars, shall be equal to the rectangle contained by the segments of any other of them.*

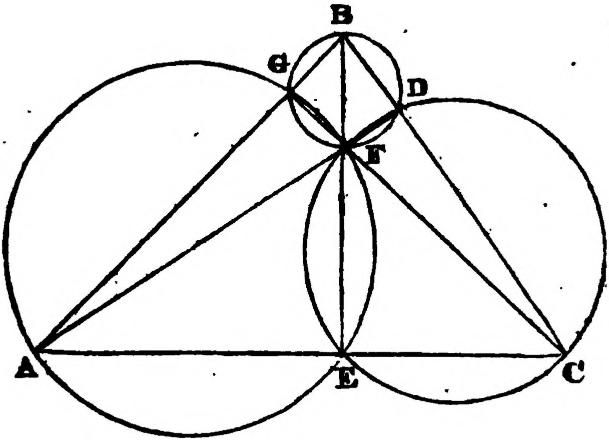
From the angles A and B, of the acute-angled $\triangle ABC$, let there be drawn AD perpendicular to BC, and BE perpendicular to AC, cutting one another in F: If, therefore, CF be drawn and produced to meet AB in G, CG (Supp. xxvi. 3.) will be perpendicular to AB:

$$\overline{AB} \times \overline{BG} = \overline{CB} \times \overline{BD}, \quad \overline{AC} \times \overline{CE} = \overline{BC} \times \overline{CD},$$

$$\text{and } \overline{BA} \times \overline{AG} = \overline{CA} \times \overline{AE};$$

$$\text{also } \overline{AF} \times \overline{FD} = \overline{CF} \times \overline{FG} = \overline{BF} \times \overline{FE}.$$

Upon *AF*, *BF*, *CF*, as diameters, describe the



circles *AEFG*, *BGFD*, *CDFE*, of which, because the angles *E*, *D*, *G*, are right angles, the circumferences will pass through *G* and *E*, *G* and *D*, *D* and *E*, respectively.

Also (E. xxxvi. 3. *Cor.*)

$$AB \times BG = \overline{EB} \times BF = CB \times BD;$$

$$\text{and } AC \times CE = \overline{GC} \times CF = BC \times CD,$$

$$BA \times AG = DA \times AF = CA \times AE.$$

Again, the △s *CFE*, *BFG*, are similar,

therefore (E. iv. 6.) $CF : FE :: BF : FG$;

therefore (E. xvi. 6.) $CF \times FG = BF \times FE$.

And, in like manner, it may be shewn that

$$BF \times FE = DF \times FA.$$

Wherefore, $AF \times FD = CF \times FG = BF \times FE$.

Cor. Since the △s *DAC*, *FAE*, are similar, as are, also, the △s *FAE*, *FBD*, therefore the △*DAC* is similar to the triangle *FBD*;

therefore (E. iv. 6.) $CD : DA :: FD : DB$,

$$\therefore\ CD \times DB = AD \times DF.$$

Again, it may, in like manner, be shewn that the $\triangle$s CGB, GFA are similar;

$$\therefore CG : BG :: AG : GF;$$

$$\therefore CG \times GF = BG \times \overline{GA}.$$

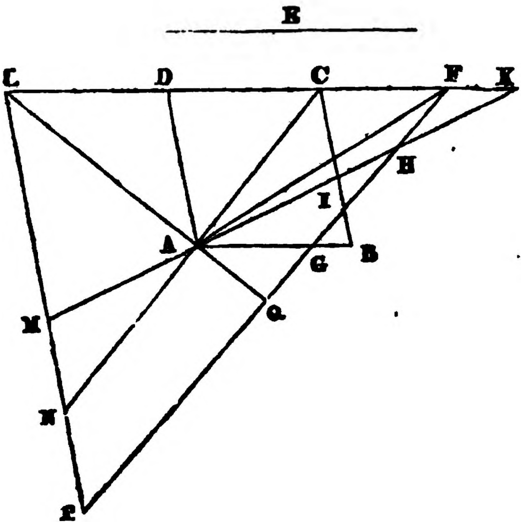
And it may, in like manner, be shewn that

$$CE \times \overline{EA} = BE \times EF.$$

PROP. III.

4. PROBLEM. *From any of the angles of an equilateral parallelogram to draw a straight line, cutting the two lines, which contain the opposite angle, so that its segment, included between those two lines, may be equal to a given finite straight line.*

Let $ABCD$ be the given equilateral parallelogram,



and E the given finite straight line: It is required to draw, from A, a straight line cutting BC, and

CD produced, so that the part of it, between these two lines, shall be equal to E.

Find (Supp. lxxiv. 1.) a square which shall be equal to $\overline{AB}^2+\overline{E}^2$; from the centre A, at a distance equal to one of the sides of this square, describe a circle cutting DC produced in F; draw the diagonal CA; from F draw FG parallel to CA, meeting AB in G; from A as a centre, at a distance equal to E, describe a circle[*] cutting FG in H; join A, H, and let AH produced meet CB in I, and DC produced in K: $IK=E$. For join A, F; produce CD to L, so that $DL=DC$; through L draw LP parallel to DA, meeting KA, CA and FG, produced, in M, N, and P, respectively; also, draw LA and produce it to meet FP in Q. And since $LD=DC$, and that LP is parallel to DA, and CB, (E. x. 6.) $MA=AI$, and $NA=AC$; also, since $AD=DL=DC$, the $\angle LAC$ is an angle in a semi-circle, and is, therefore, (E. xxxi. 3.) a right angle; therefore (*constr.* and E. xxix. 1.) the $\angle$s LQP, LQF, are right angles: and since LQ bisects NC, it bisects (Supp. lxi. 1.) FP: Therefore FP is the double of QP; and (E. xii. 2.)

$$\overline{AF}^2=\overline{AH}^2+\overline{HF}^2+2\,\overline{HF}\times\overline{HQ}$$
$$=AH^2+\overline{FH}\times\overline{HP};$$

but (*constr.*)

$$AF^2=\;E^2\;+\overline{AC}^2=AH^2+AC^2;$$
$$\therefore\; AC^2=FH\times HP:$$

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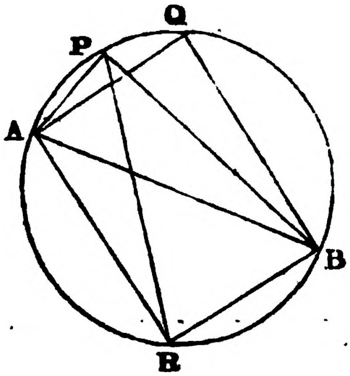
[*] Since (E. xii. 2.) $AF^2>(\overline{AG}^2+\overline{GF}^2)$, the $\angle AGF$ being necessarily obtuse, therefore (*constr.*) $(E^2+GF^2)>(AG^2+GF^2)$; $\therefore\; E^2>AG^2$, and $E>AG$, and $E<AF$; wherefore, the circle described from A as a centre, at a distance equal to E, will cut FG, between F and G.

and (E. xvii. 6.) $PH : AC :: AC : HF$;
but (E. iv. 6.) $PH : AC$ or $AN :: HM : AM$,
and $AC : HF :: AK : HK$;
therefore (E. xi. 5.) $HM : AM :: AK : HK$;
therefore (*dividendo*)
$AH : MA :: AH : HK$;
therefore (E. xiv. 5.) $MA = HK$; but $MA = AI$;
$\therefore AI = HK$, and $AH = IK$; but (*constr.*) $AH = E$;
$\therefore IK = E$.

PROB. IV.

5. THEOREM. *If from any two points, in the semi-circumference of a given circle, straight lines be drawn to the extremities of the diameter, which subtends the semi-circumference, the rectangle contained by the two, that are terminated in one extremity of the diameter, together with the rectangle contained by the other two, shall be less than the square of the diameter.*

Let the semi-circumference $\overset{\frown}{APB}$, of the given circle $APBR$, be subtended by the diameter $\overline{AB}$,



and from any two points P, Q in $\overset{\frown}{APB}$, let there be drawn to A and B, $\overline{PA}$, $\overline{QA}$, $\overline{PB}$, $\overline{QB}$:

$$\overline{AP}\times\overline{AQ}+\overline{BP}\times\overline{BQ}<\overline{AB}^2.$$

In the semi-circumference ARB place $\overline{BR}$ equal to AQ, and join A, R, and P, R; wherefore (E. xxviii. 3.) $\overset{\frown}{BR}=\overset{\frown}{AQ}$; $\therefore$ $\overset{\frown}{AR}=\overset{\frown}{QB}$, and (E. xxix. 3.) $\overline{AR}=BQ$. But (Supp. lvii. 6.)

$$AP\times BR+PB\times\overline{AR}=AB\times PR,$$

that is, $AP\times AQ+BP\times BQ=AB\times PR$.

And, since $\overset{\frown}{BR}=\overset{\frown}{AQ}$, add to both $\overset{\frown}{AR}$, and

$$\overset{\frown}{ARB}\ (=\overset{\frown}{QAR})>\overset{\frown}{PAR};$$

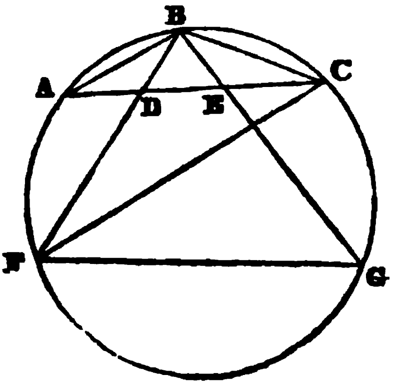
therefore (Supp. xliv. 3.) $AB>PR$;

$$\therefore\ AP\times AQ+BP\times BQ<AB^2.$$

Prop. V.

6. Theorem. *If from the bisection of any given arch of a circle, two straight lines be drawn, cutting the chord of the arch and the circumference, the four points of intersection shall also lie in the circumference of a circle.*

From the bisection B, of the arch ABC, of the



circle $ABCF$, let there be drawn any two straight lines BF, BG, cutting $\overline{AC}$ in D and E, and the circumference of $ABCF$ in F and G: the four points D, E, G, F, shall also lie in the circumference of a circle.

For join A, B; B, C; C, F; and F, G: then (E. xxxii. 1.) the $\angle BED = \angle BCE + \angle CBG =$ (E. xxvii. 3. and E. xxi. 3.)

$$\angle BFC + \angle CFG = \angle BFG;$$

and the $\angle FBG$ is common to the two $\triangle$s BDE, BGF;

therefore (E. iv. 6.) $FB : BG :: EB : BD$;

therefore (E. xvi. 6.) $\overline{FB} \times \overline{BD} = \overline{GB} \times \overline{BE}$;

therefore (Supp. Part. II. xi. 1.) the four points F, D, E, G, are in the circumference of a circle.

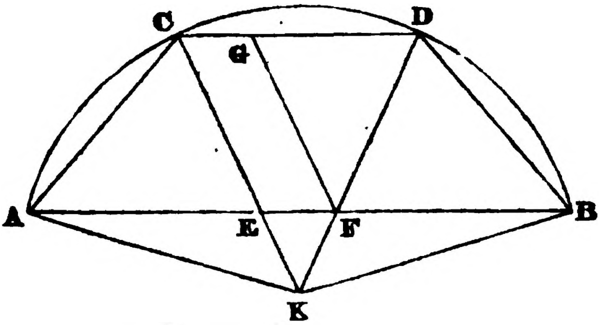
Prop. VI.

7. **Theorem.** *If a given arch of a circle be divided into three equal parts, their chords shall, together, exceed the chord of the whole arch, by a straight line, which is to the chord of one of the equal parts as the square of that chord is to the square of the semi-diameter.*

Let $\overset{\frown}{ACDB}$ be an arch of the circle, of which the centre is K, and let it be divided into three equal parts $\overset{\frown}{AC}$, $\overset{\frown}{CD}$, $\overset{\frown}{DB}$; $\overline{AC} + \overline{CD} + \overline{DB}$ shall exceed $\overline{AB}$ by a straight line which is to $\overline{CD}$ as $\overline{CD}^2$ is to $\overline{KD}^2$.

Join A, K, and C, K, and B, K. Let CK and DK cut AB in E and F respectively, and from F draw FG parallel to KC, cutting $\overline{CD}$ in G.

Then (E. xx. 3.) $\angle ABD$ or $FBD = \frac{1}{2} \angle AKD = DKB$; and the $\angle KDB$ is common to the two



$\triangle$s FBD, DKB; therefore $\angle BDK (= \angle KBD) = \angle BFD$, and (E. vi. 1.) $BD = BF$.

Also (E. iv. 6.) $DF : DB :: DB : DK$;

$$\text{therefore } DF : DK :: \overline{DB}^2 \text{ or } \overline{DC}^2 : \overline{DK}^2,$$

$$\text{and } DF : DK :: DG : DC;$$

therefore (E. xvi. 5.)

$$DG : DC :: \overline{DC}^2 : \overline{DK}^2.$$

But, since $BF = BD$, and in like manner $AE = AC$, and since (*constr.* and E. xxxi. 1.) $EF = CG$, it is manifest that GD is the excess of $\overline{AC} + \overline{CD} + \overline{DB}$ above AB.

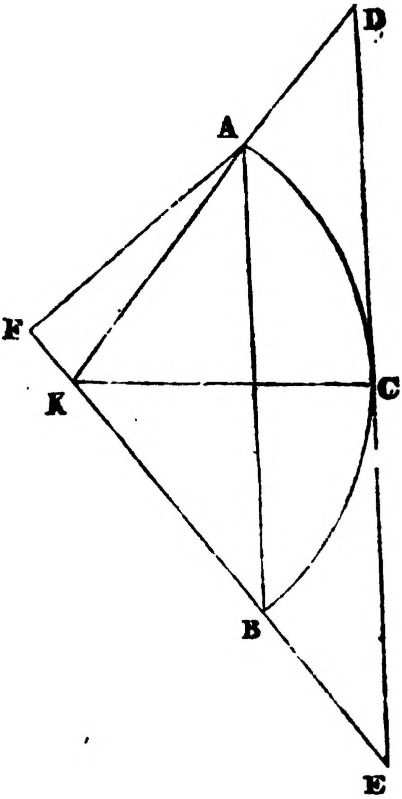
8. SCHOLIUM. If the chord $\overline{AB}$, of an arch $\overset{\frown}{ACB}$ of a circle be given, and if it be required to find, algebraically, the chord of $\frac{1}{3} \overset{\frown}{ACB}$, let $a = \overline{AB}$, $r = AK$, and $x =$ the chord of $\frac{1}{3} \overset{\frown}{ACB}$; therefore, by the above theorem, $3x = a + \frac{x^3}{r^2}$, and the determination of the value of x is thus made to depend upon the solution of the cubic equation

$$x^3 - 3r^2x + ar^2 = 0.$$

Prop. VII.

9. **Theorem.** *The side of an equilateral polygon, inscribed in a circle, is a mean proportional between the side of a similar polygon described about it, and the half of the side of an equilateral polygon of half the number of sides inscribed in the circle.*

In a circle, of which K is the centre, let $\overline{AB}$ be



the side of an inscribed equilateral polygon; bisect (E. xxx. 3.) $\overset{\frown}{ACB}$ in C, and join K, A, and K, B; through C draw $\overline{DCE}$ touching the circle in C, and meeting KA and KB produced in D and E;

and from A draw AF at right angles to KB. DE (E. xxx. 3. E. xviii. 3. and E. xxviii. 1.) is parallel to AB, and it is the side of the circumscribed similar polygon, of the same number of sides; and AF is the half of the side of an inscribed equilateral polygon of half the number of sides.

Then, $DE : AB :: AB : AF$.

For the △s DKE, AKB, are similar, as are, also, the right-angled △s KCE, AFB, the ∠ KEC being (*constr.* and E. xxix. 1.) equal to the ∠ FBA;
therefore (E. iv. 6.) $DE : AB :: KE : KB$ or KC,
and $KE : KC :: BA : AF$,
therefore (E. xi. 1.) $DE : AB :: AB : AF$.

10. **Scholium.** Hence it is manifest, that the chord of any arch is a mean proportional between the sine of the arch, and the double of the tangent of half the arch. Or, if A be any arch,

$$\text{sin. } 2A : 2 \text{ sin. } A :: 2 \text{ sin. } A : 2 \text{ tang. } A;$$

$$\text{therefore sin. } 2A = \frac{2 \text{ sin}^2. A}{\text{tang. } A}.$$

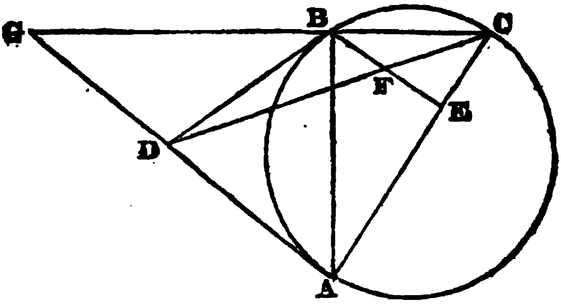
Prop. VIII.

11. **Theorem.** *If two tangents be drawn to a circle, from any given point without it, and if, from either of the two points of contact, there be drawn a perpendicular to that diameter, which passes through the other point of contact, the perpendicular shall be bisected by the straight line joining the given point and the end of the diameter.*

From the point D, without the circle ACB, let there be drawn the tangents DA, DB; also, from A, let there be drawn the diameter AC, and from B, BE

at right angles to AC; and let D, C be joined: DC shall bisect BE.

For let DC cut BE in F; join A, B, and C, B,



and since (*constr.*) CAD is a right angle, and (E. xxxi. 3.) ACB is less than a right angle, AD and CB being produced far enough, will meet; let them meet in G. Then (E. xxxvi. 3. *Cor.*) $DA = DB$; therefore (E. v. 1.) the $\angle ABD = \angle DAB$;

$\therefore \angle BAD + \angle DBG = \angle ABD + \angle DBG = \angle ABG$; but (E. xxxi. 3.) $\angle ABC$ is a right angle; wherefore (E. xiii. 1.) $\angle ABG$ is also a right angle;

$$\therefore \angle ABG = \angle ABC;$$

and (E. xxxii. 1.) $\angle ABC = \angle BAG + \angle G$;

$$\therefore \angle BAD + \angle DBG = \angle BAD + \angle G;$$

$$\therefore \angle DBG = \angle G;$$

and (E. vi. 1.) $DB = DG$; but $DB = DA$;

$$\therefore DG = DA.$$

Again, (*constr.* and E. xxviii. 1.) BE is parallel to GA; therefore (E. iv. 6.)

$$CD : CF :: DG : BF :: DA : FE;$$

and $DG = DA$; therefore (E. xiv. 5.) $BF = FE$.

Cor. Since (E. iv. 6.)

$$CE : EB :: CA : AG \text{ or } AD + DB,$$

therefore (E. xvi. 5.)

$$CE : CA :: EB : AD + DB.$$

And since, by continually bisecting the arch *AB*, the ratio of *CE* to *CA* may be made to approximate, indefinitely, to a ratio of equality, it is plain that a regular polygon of an even number of sides may be inscribed in a given circle, and a regular polygon of double that number of sides described about it; the perimeters of which polygons shall have a difference less than any given finite straight line.

12. SCHOLIUM. If v be the versed sine, and s the sine of any arch (A) of a circle, the diameter of which is $2r$, it is manifest from the corollary, that

$$2r - v : 2r :: s : 2 \text{ tang. } \frac{A}{2};$$

$$\text{wherefore, tang. } \tfrac{1}{2}A = \frac{rs}{2r - v}.$$

PROP. IX.

13. THEOREM. *The rectangle contained under the tangent and the sine, of any arch of a circle, is greater than the square of the chord of the arch**.

Let $\overset{\frown}{AB}$ be any arch of the circle *ABC*, of which $\overline{AB}$ is the chord, the diameter *AC*, and *BD* at right angles to it; and let *BE*, drawn to touch the circle in *B*, meet *CA* produced in *E*: $\overline{EB} \times \overline{BD} > \overline{AB}^2$.

Join *B*, *C*: (E. xxxii. 3.) $\angle EBA = \angle ACB$, and the $\angle BEC$ is common to the two $\triangle$s *EAB*, *EBC*; wherefore they are equiangular, and (E. iv. 6.)

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* The terms of Trigonometry are used in enunciating this theorem, merely for the sake of conciseness. If the reader be not already acquainted with them, the particular enunciation which follows, will sufficiently explain the meaning of the proposition.

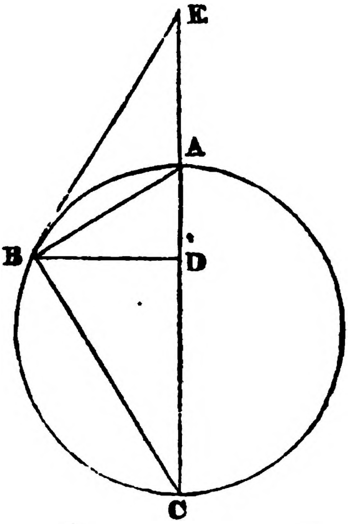
$$EB : BA :: EC : CB,$$

also (E. viii. 6.)

$$BD : BA :: CD : CB;$$

therefore (Supp. i. 6.)

$$\overline{EB}\times\overline{BD} : \overline{BA}^2 :: \overline{EC}\times\overline{CD} : \overline{CB}^2;$$



but (E. viii. 6. and E. xvii. 6.) $\overline{AC}\times\overline{CD}=\overline{CB}^2$;

$$\therefore \overline{EC}\times\overline{CD} > \overline{CB}^2; \quad \therefore \overline{EB}\times\overline{BD} > \overline{BA}^2.$$

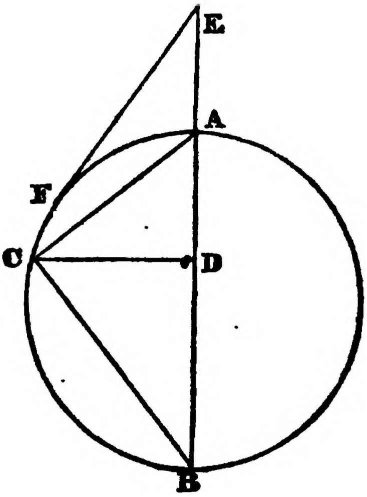
14. **Scholium.** Hence it may readily be shewn, that if the △ *EBD*, and the circle *ABC* revolve about *EC*, the convex surface of the cone, generated by the revolution of the triangle, is greater than the surface of the spherical segment included within the cone.

Prop. X.

15. **Theorem.** *If the diameter, passing through the extremity of a given arch of a circle, be produced,* [illegible] *that the part produced shall be equal to the versed sine of the given arch, and if, from the end of the*

part produced, a tangent be drawn to the circle, its square shall exceed the square of the versed sine, by the square of the chord of the arch.

Let the diameter *AB*, of the circle *ABC*, be produced to *E*, so that *AE* shall be equal to *AD*, the versed sine of the given arch $\overset{\frown}{AC}$, (of which the chord is $\overline{AC}$) determined by drawing *CD* perpendicular to *AB*; and from *E* let *EF* be drawn to touch the circle in *F*:



$$\overline{EF}^2 = \overline{AC}^2 + \overline{AD}^2.$$

Join *C*, *B*: then (E. xxxvi. 3.)

$$\overline{EF}^2 = \overline{BE} \times \overline{EA} = (\mathit{constr.})\ (\overline{BA} + \overline{AD}) \times \overline{AD}$$

$$= (\text{E. i. 2.})\ \overline{BA} \times \overline{AD} + \overline{AD}^2;$$

but (E. viii. 6. and E. xvii. 6.)

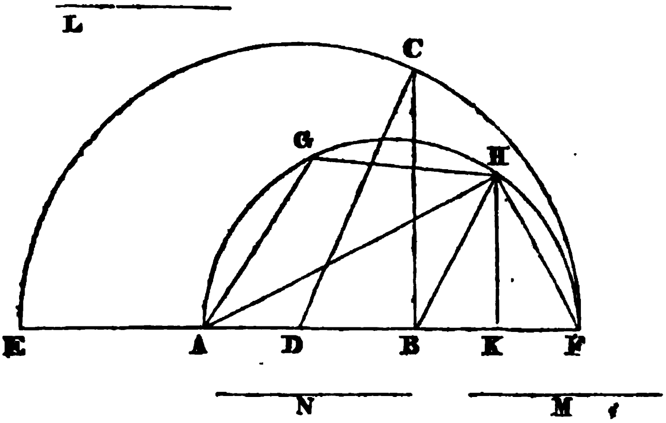
$$\overline{BA} \times \overline{AD} = \overline{AC}^2; \quad \therefore \overline{EF}^2 = \overline{AC}^2 + \overline{AD}^2.$$

Cor. $EF^2 + \overline{CD}^2 = 2\overline{AC}^2$.

PROP. XI.

16. **PROBLEM.** *Three sides of a quadrilateral rectilineal figure, of which two are equal to one another, being given, to construct it, so that the fourth side shall be the diameter of a circle described about the figure.*

Let L, M, N, be the three given sides, of which $L=M$: It is required to construct, &c.



Take $\overline{AB}=N$; find (Supp. lxxiv. 1.) a square which shall be the double of L^2 or of M^2; draw BC at right angles to AB, and make it equal to the side of that square; bisect AB in D, and join D, C; from the centre D, at the distance DC, describe the semi-circle ECF; also, upon AF, as a diameter, describe the semi-circle AHF; place in it $\overline{AG}$ equal to AB or N, bisect $\overset{\frown}{GHF}$ in H, and join G, H, and H, F; $AGHF$ is the figure which was to be constructed.

For, from H draw HK perpendicular to AF, and join A, H, and B, H. Then (*constr.* and E. xxvii. 3.)

the $\angle GAH = \angle HAF$ or HAB, $\overline{GA} = AB$,

and AH is common to the $\triangle$s AGH, ABH;

therefore (E. iv. 1.) $\overline{GH} = HB$;

also (E. xxix. 3.)

$\overline{GH} = \overline{HF}$; $\therefore HB = \overline{HF}$, and $\overline{HB}^2 = \overline{HF}^2$:

also (*constr.* and E. xlvii. 1.)

$\overline{BH}^2 = \overline{BK}^2 + \overline{KA}^2$, and $\overline{HF}^2 = \overline{FK}^2 + \overline{KH}^2$;

$\therefore \overline{BK}^2 = \overline{FK}^2$, and $BK = FK$.

Wherefore, $\overline{AF} \times \overline{FB} = 2\,\overline{AF.FK}$.

Again, since $ED = DF$, and $AD = DB$,

therefore $EA = BF$, and $EB = AF$;

$\therefore \overline{EB} \times \overline{BF} = \overline{AF} \times \overline{FB}$

$=$ (as has been shewn) $2\,\overline{AF} \times \overline{FK}$:

But (E. viii. 6. and E. xvii. 6.) $\overline{EB} \times \overline{BF} = \overline{BC}^2$,

and $\overline{AF} \times \overline{FK} = \overline{FH}^2$, $\therefore \overline{BC}^2 = 2\overline{FH}^2 =$ (*constr.*) $2L^2$;

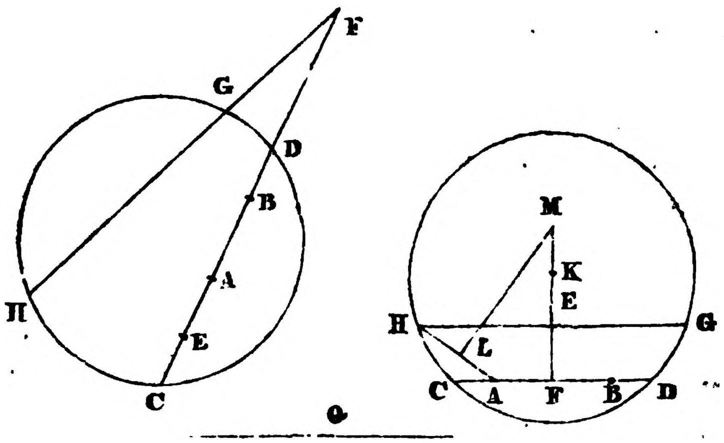
$\therefore \overline{FH}^2 = L^2$, and $\overline{FH} = L = \overline{HG} = M$;

and AG was made equal to N. Therefore $AGHF$ is the figure which was to be constructed.

PROP. XII.

17. PROBLEM. *To describe the circumference of a circle, which shall pass through two given points, within a given circle, and shall cut the circumference of that circle in two points, the distance of which shall be equal to a given straight line, not greater than a diameter of the given circle.*

Let A, B be two points in the circle $HGDC$, and Q a straight line, not greater than a diameter of $HGDC$: It is required to describe a circle, the circumference of which shall pass through *A, B,* and



shall cut the circumference of *HGDC* in two points, the distance of which shall be equal to *Q*.

Join *A, B,* and produce *AB* to meet the circumference again in *C, D*. And, first, if $CA = DB$, find the centre *K*, of *HGDC*; draw (Supp. Part II. vii. 1.) the chord *HG* equal to *Q*, and parallel to *CD*; from *K* draw *KEF* perpendicular to *HG* and *CD*, which (E. iii. 3.) bisects *HG* in *E*, and *CD* in *F*; and, since $CA = BD$, $AF = FB$; join *A, H*, bisect *AH* in *L*, and from *L* draw *LM* at right angles to *HA*, meeting *KF* in *M*; wherefore (*constr.* and E. iv. 1.) *M* is equidistant from *H, A, B* and *G*; and a circle described from the centre *M*, at the distance *MH*, will pass through *A* and *B*, and cut *HCDG* in *G*, so that $HG = Q$.

But if *CA* be not equal to *BD*, one of them is the greater; let *CA* be greater than *BD*, and from *CA* cut off *AE* equal to *BD*; also, from *CD*, produced, cut off *DF* equal to a fourth proportional (E. xii. 6.)

to *CE*, *EA*, and *AD*, and from *F* draw $\overline{FGH}$ so that (Supp. Part II. ix. 1.) *GH* may be equal to *Q*; then if (E. v. 4.) a circle be described, the circumference of which shall pass through *H*, *G*, *B*, it will also pass through *A*.

For (*constr.*) $CE : EA :: AD : DF$;
∴ (*componendo*) $CA : AE$ or $BD :: AF : FD$;
and (*alternendo*) $CA : AF :: BD : DF$;
∴ (*componendo*) $CF : AF :: BF : FD$;
therefore (E. xvi. 6.) $\overline{CF} \times \overline{FD} = \overline{AF} \times \overline{FB}$;
but (E. xxxvi. 3. *Cor.*) $CF \times FD = \overline{HF} \times \overline{FG}$;
$\therefore HF \times FG = AF \times FB$,

and (Supp. Part II. xi. 1.) the circumference of the circle, which passes through *H*, *G*, *B*, shall also pass through *A*.

PROP. XIII.

18. **PROBLEM.** *In the given straight line joining the centres of two circles, that lie wholly without one another, to find a point, such that if tangents be drawn from it to the two circles, the perpendiculars drawn from the points of contact, to that given line, shall cut off from the diameters two segments, which shall be to each other, as the diameters are to each other.*

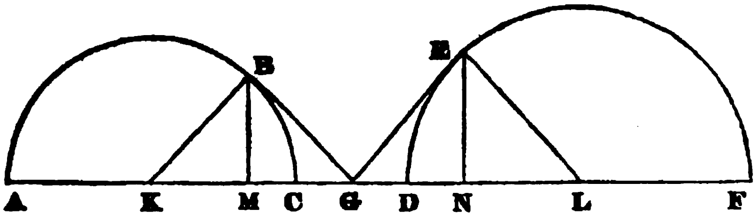
Let the given straight line *AKLF*, pass through the centres *K* and *L*, of the circles *ABC*, *DEF*: It is required, &c.

Divide (E. x. 6.) *CD* in *G*, so that

$$CG : GD :: KC : DL.$$

Then is *G* the point which was to be found.

From *G* draw *GB* and *GE* touching *ABC*, *DEF* in



B and *E* respectively; draw *BM* and *EN* perpendicular to *ACDF*, and join *B*, *K* and *E*, *L*.

Then since (*constr.*)

$$CG : GD :: CK : DL,$$

(*permutando* and *componendo*)

$$GK : KC :: GL : LD;$$

but (E. viii. 6.)

$$GK : KB \text{ or } KC :: KC : KM,$$

$$\text{and } GL : LD :: LD : LN;$$

therefore (E. xi. 5.)

$$KC : KM :: LD : LN,$$

and (*convertendo*)

$$KC : MC :: LD : ND;$$

therefore (*permutando*)

$$KC : LD :: MC : ND,$$

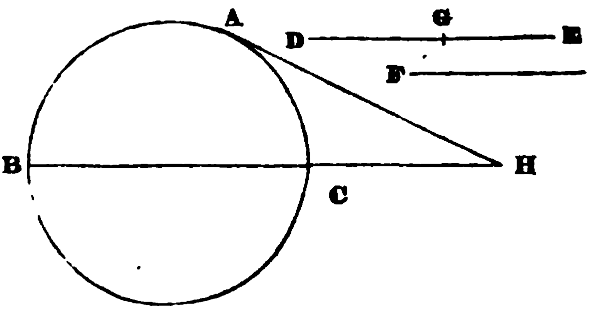
and (E. xv. 5.)

$$AC : DF :: MC : ND.$$

PROP. XIV.

19. PROBLEM. *To draw a tangent to a given circle, to meet a given diameter produced, so that the tangent shall have a given ratio of inequality to the part produced of the diameter, which is terminated by the tangent.*

Let *BC* be the diameter of the circle *ABC*, and



DE and *F* two unequal straight lines, of which *DE* is the greater: It is required to draw a tangent to *ABC*, meeting *BC* produced, so that it shall be to the part produced of *BC* as *DE* is to *F*.

From *ED* cut off *EG* equal to a third proportional to *DE* and *F*; produce *BC* to *H*, so that *CH* shall be equal to a fourth proportional to *DG*, *GE* and *BC*; and from *H* draw *HA* touching *ABC* in *A*: then is *AH* the tangent which was to be drawn.

For since (*constr.*)

$$BC : CH :: DG : GE,$$

(*componendo*)

$$BH : CH :: DE : GE,$$

but (E. xxxiv. 3, and E. xvii. 6.)

[illegible]$H : CH :: \overline{AH}^2 : \overline{CH}^2$

also (*constr.* and E. xvii. 6.)

$$DE : GE :: \overline{DE}^2 : F^2;$$

therefore (E. xi. 5.)

$$\overline{AH}^2 : \overline{HC}^2 :: \overline{DE}^2 : F^2,$$

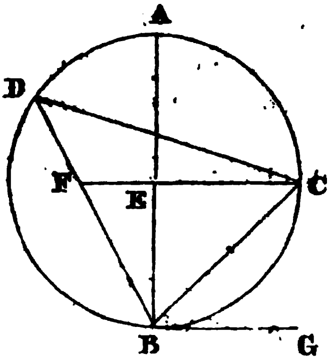
therefore (Supp. i. 6. *Cor.* 2.)

$$AH : HC :: DE : F.$$

PROP. XV.

20. THEOREM. *If two unequal chords of a circle be terminated in the same given point, and if, from the other extremity of either of them, there be drawn a perpendicular to the diameter, which terminates in the given point, that chord shall be a mean proportional between the other given chord, and the segment cut off from it, by the perpendicular.*

Let the two unequal chords *BC*, *BD*; of the



circle *ABCD*, be terminated in the same point *B* of the circumference; let *AB* be a diameter, drawn from *B*; and from the other extremity of either of the chords, as from *C*, let there be drawn *CE* perpendicular to *AB*, and meeting the other chord *BD* in *F*: then

$$BD : BC :: BC : BF.$$

For, through *B*, let *BG* be drawn at right angles to *AB*, and therefore (E. xxviii. 1.) parallel to *CF*, and (E. xvi. 3.) touching the circle in *B*. Wherefore, (E. xxix. 1. and E. xxxii. 3.)

$$\angle GBC = \angle BCF = \angle BDC;$$

and the $\angle DBC$ is common to the two $\triangle$s BDC, BCF; therefore, they are equiangular, and (E. iv. 6.)

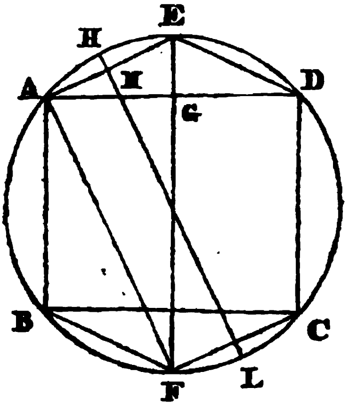
$$DB : BC :: BC : BF.$$

The proof is the same, when the two given chords lie on the same side of the diameter, which terminates in their point of concourse.

PROP. XVI.

21. THEOREM. *The square of the side of an equilateral and equiangular octagon, inscribed in a given circle, and the square of the diameter of the circle, have no common measure.*

In the given circle $ABFCDE$, inscribe (E. iv. 4.)



the square $ABCD$: draw the diameter EGF at right angles to $\overline{AD}$, and cutting it in G, and join A, E: $\overline{AE}$ is, manifestly, the side of a regular octagon, inscribed in the circle, and AE^2 and $\overline{EF}^2$ have no common measure.

For, join F, A; the $\angle EAF$ (E. xxxi. 3.) is a right angle, and (*constr.*) AG is at right angles to EF,

therefore (E. viii. 6.) $FE : EA :: EA : EG$;

$$\therefore EF^2 : \overline{EA}^2 :: EF : EG.$$

But $EG = \frac{1}{2}(EF - AB)$; and EF is a diameter of the circle, and AB a side of the inscribed square; wherefore EF and AB are incommensurable, and [*]$\frac{1}{2}(EF - AB)$, or EG, and EF are incommensurable. Therefore, also, (E. x. 10.) EF^2 and EA^2 are incommensurable.

22. Cor. 1. Hence, and from E. ix. 10. it is plain, that the diameter of a circle, and the side of an inscribed regular octagon are incommensurable.

23. Cor. 2. If the diameter HL be drawn at right angles to the side AE of the inscribed regular octagon, and if A, H be joined, it may, in like manner, be shewn that $\overline{AH}^2$ and $\overline{HL}^2$ are incommensurable, and so on.

---

[*] If A and B be incommensurable, $A \pm B$ and A are so.

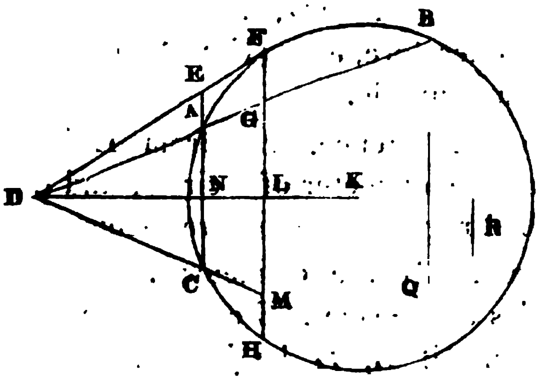
For, if it be possible, let $\frac{A+B}{D} = p$, and $\frac{A}{D} = q$; wherefore, $\frac{A}{D} \sim \frac{A \pm B}{D} = q \sim p$, and is an integer, that is, $\frac{B}{D}$ is an integer, as is also $\frac{A}{D}$; so that A and B have a common measure, which is contrary to the supposition. And it is manifest, that if $A \pm B$ and A be incommensurable, so will be

$$m \cdot (A \pm B), \text{ and } A, \text{ and } \frac{1}{n}(A \pm B) \text{ and } A.$$

PROP. XVII.

24. PROBLEM. *In a given circle, to draw a chord parallel to a given straight line, so as that it shall be divided, by another given chord, into two parts, which shall be to one another in a given ratio of inequality.*

Let *ABC* be the given circle, *Q* the given straight line, *AB* the given chord: It is required to draw a chord in *ABC*, which shall be divided by *AB* into two parts, which shall be to one another in the given ratio of *Q* to *R*.



From *A* draw *AC* parallel to *Q*; find the centre *K*, and from *K* draw *KD* perpendicular to *AC*, meeting *AC* in *N*, and *BA* produced in *D*; produce *CA* to *E*, so that *AE* may be equal to a fourth proportional to *Q*–*R*, *R*, and *CA*; join *D*, *E* and produce it to * meet the circumference in *F*, and from F draw the chord $\overline{FH}$ parallel to EC or to Q, and cutting AB in G: then

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* If *DE*, produced, does not meet the circle, the problem is impossible. For the given ratio of inequality has manifestly a limit: And if *Q* be perpendicular to *AB*, the point must be determined as before, and *EF* must be drawn parallel to *AB*.

$$HG : GF :: Q : R.$$

Join D, C; and let FH meet KD in L, and DC, produced, in M. Then

(E. iv. 6.) $LG : NA :: LD : ND :: LM : NC$;
therefore (E. xiv. 5.)

$$LG : NA :: LM : NC;$$

but (E. iii. 3.) $NA = NC$; therefore (E. ix. 5.) $LG = LM$; also (E. iii. 3.) $LF = LH$; $\therefore$ $GF = MH$, and $\therefore$ $GH = MF$; but (*constr.* and E. iv. 6.)

$$MF : GF :: CE : AE,$$

that is, $HG : GF :: CE : AE$;
but (*constr.*) $Q - R : R :: CA : AE$;
therefore (*componendo*)

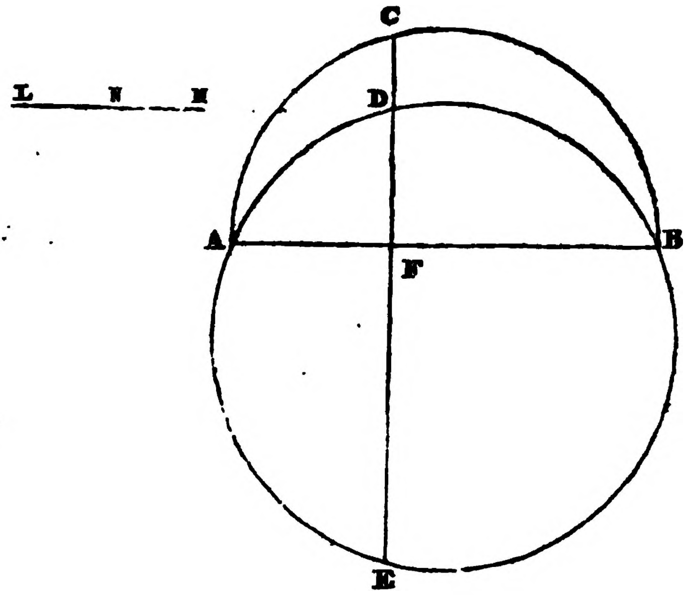
$$Q : R :: CE : AE;$$

$$\therefore HG : GF :: Q : R.$$

PROP. XVIII.

25. PROBLEM. *To find a point, in the circumference of a given semi-circle, from which if a perpendicular be drawn to the diameter, it shall be divided, by the arch of another circle, which has the semi-circle's diameter for its chord, similarly to a given divided straight line.*

Let ACB be the semi-circle; let its diameter AB be the chord of an arch $\overset{\frown}{ADB}$, of another circle; and let LM be the given straight line divided into two parts in N: It is required to find a point in $\overset{\frown}{ACB}$, from which, if a perpendicular be drawn to,



AB, it shall be divided by $\overset{\frown}{ADB}$, similarly to *LM*.

Complete the circle *ADBE*, and in it (Supp. Part II. 3.) draw the chord *DFE*, such that

$$EF : FD :: \overline{LM}^2 : \overline{LN}^2;$$

and produce *FD* to *C*: *C* is the point required.

For (E. i. 6. and E. xi. 5.)

$$\overline{EF} \times \overline{FD} : FD^2 :: LM^2 : LN^2:$$

that is, (E. xxxv. 3.)

$$AF \times FB \text{ or } FC^2 : FD^2 :: LM^2 : LN^2;$$

$$\therefore FC : FD :: LM : LN,$$

and (*dividendo*)

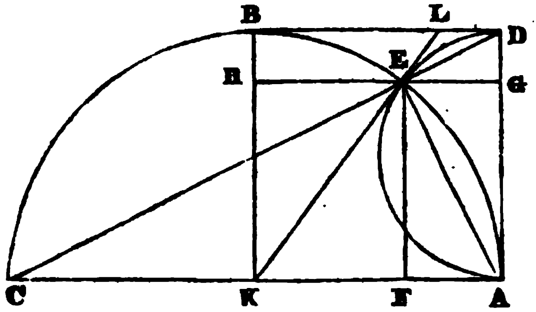
$$CD : DF :: MN : NL.$$

PROP. XIX.

26. **PROBLEM.** *To divide the quadrant of the circumference of a circle, into two arches, such that*

the aggregate of their versed sines shall be equal to the right sine of the less.

Let $\overset{\frown}{AB}$ be a quadrant of the circumference of



the circle *ABC*, of which *K* is the centre, and *AKC* a diameter, it is required, &c.

From *A* draw *AD* at right angles to *CA*, and make it equal to *KA*; join *C*, *D*, and let *CD* cut the circumference in *E*: *E* is the point of division, which was to be found.

Join *K*, *B*; and through *E* draw *GEH* parallel to *AC*, and, therefore, perpendicular to *BK*, and *EF* parallel to *DA*, and therefore perpendicular to *CA*. Therefore, *BH* is the versed sine, and *EH* is the sine of $\overset{\frown}{EB}$, and *FA* is the versed sine of $\overset{\frown}{EA}$. And, since the △s *CAD*, *EGD*, are similar,

(E. iv. 6.) $EG : GD :: CA : AD :: 2 : 1$;

$\therefore EG = 2\,GD$; that is, (E. xxxiv. 1.) $AF = 2\,BH$;

Again, join *E*, *A*; and since (E. xxxi. 3.) the angle *CEA* is a right angle, the ∠*AED* is also a right angle,

and (E. viii. 6.) $AG : GE :: GE : GD :: 2 : 1$;

$\therefore AG = 2\,GE$; that is, $HK = 2\,AF = 4\,BH$;

and $\therefore BK$ or $KA = 5\,BH$; $\therefore BH + AF = 3\,BH$; also, $EH = KF = KA - AF = 5\,BH - 2\,BH = 3\,BH$;

$$\therefore BH + AF = EH.$$

27. Cor. 1. By the same construction, an arch (BE) may be found, such that the aggregate of its sine and versed sine ($EH + HB$) shall be equal to its cosine (HK).

28. Cor. 2. By the same construction, also, a quadrant of the circumference of a circle is divided into two arches $\widehat{BE}$, $\widehat{EA}$, such that the aggregate of the sine and versed sine of the less shall be equal to the sine, or to the double of the versed sine of the greater; for

$$BH + HE = EF = 2\,AF.$$

29. Cor. 3. The five lines BH, AF, EH, HK, KA, are arithmetic proportionals.

Prop. XX.

30. Problem. *To divide the quadrant of the circumference of a circle into two arches, such that the aggregate of the tangent, and the secant of the lesser arch, shall be equal to the diameter of the circle.*

Let $\widehat{AB}$[*] be a quadrant of the circumference of the circle ABC; it is required, &c.

The same construction being made, as in the preceding proposition, E is the point of division, which was to be found.

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[*] See the figure to the preceding proposition.

For join D, B, and K, E, and let KE, produced, meet BD in L: wherefore, (E. xxxiii. 1. E. xxix. 1. and E. xvi. 3. *Cor.*) BL is the tangent, and KL is the secant of $\widehat{BE}$. And since the $\triangle$s KHE, KBL, are similar,

$$\text{(E. iv. 6.) } KL : LB :: KE : EH;$$

therefore (*componendo*)

$$KL+LB : LB :: KE+EH : EH;$$

$$\text{also, } LB : BK :: EH : HK;$$

therefore (*ex æquo*)

$$KL+LB : BK :: KE+EH : HK.$$

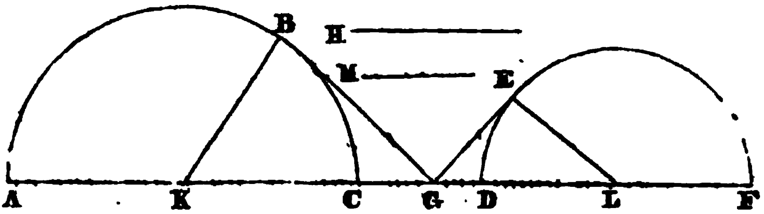
But it is manifest from the demonstration of the preceding proposition, that $(KE+EH)=2\,HK$;

$$\therefore\ (KL+LB) = 2\,BK = AC.$$

PROP. XXI.

31. PROBLEM. *In the given straight line joining the centres of two given circles, that lie wholly without each other, to find a point, such that the two tangents, drawn from it to the two given circles, shall be to one another in a given ratio.*

Let ABC, DEF, be the two given circles, and let



the straight line $AKLF$ pass through the centres K and L: It is required to find a point in CD, from which, if tangents be drawn to *ABC*, *DEF*, they shall be to one another in a given ratio, namely, in that of *AC* to *H*.

Find (E. xi. 6.) a third proportional *M*, to *AC* and *H*; in *CD* find (Supp. Part II. v. 2.) a point *G*, such that $\overline{AG}\times\overline{GC} : \overline{FG}\times\overline{GD} :: AC : M$; *G* is the point which was to be found.

From *G*, draw (E. xvii. 3.) the tangents *GB*, *GE*, to the circles *ABC*, *DEF*; then (*constr.*)

$$AC : M :: AG\times GC : FG\times GD,$$

that is, (*constr.* and E. xxxvi. 3.)

$$\overline{AC}^2 : \overline{H}^2 :: \overline{BG}^2 : \overline{EG}^2;$$

therefore (Supp. i. 6. *Cor.* 2.)

$$AC : H :: BG : EG.$$

32. Cor. If the △s *GBK*, *GEL*, are required to be in a given ratio to one another, let it be that of *AK* to a given straight line *M*: find the point *G*, such that $GB : GE :: LD : M$; then the

$$\triangle GBK : \triangle GEL :: AK : M.$$

For $\triangle GBK : \triangle GEL :: \overline{GB}\times\overline{BK} : \overline{GE}\times\overline{EL}$;

but (*constr.*) $GE\times EL = GB\times M$;

therefore (E. i. 6.)

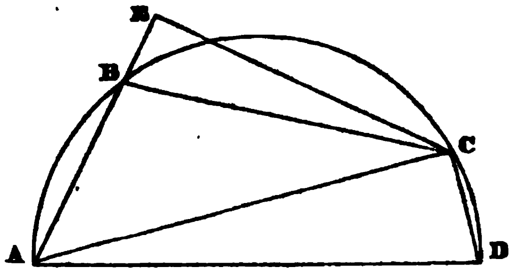
$$\triangle GBK : \triangle GEL :: BK \text{ or } AK : M.$$

Prop. XXII.

33. Theorem. *If a trapezium, inscribed in a circle, have a diameter for one of its sides, the excess of the square of the diameter above the squares of the three remaining sides shall be to the rectangle contained by the first and second of those sides, as the*

double of the third of them is to the diameter of the circle.

Let the trapezium $ABCD$, inscribed in the circle



$ABCD$, have the diameter AB, for one of its sides; then

$\overline{AD}^2 - (\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2) : AB \times BC :: 2\,CD : AD.$

Join A, C; the $\angle ABC$ (*hyp.* and E. xxxi. 3.) is obtuse; produce AB, and from C draw CE perpendicular to it, and meeting it in E. And since (E. xxii. 3.) the $\angle$s ABC, ADC, are equal to two right angles, as are, also, the two $\angle$s ABC, CBE, the $\angle ADC = \angle CBE$, and (*constr.* and E. xxxi. 3.) the $\angle$s BEC, ACD, are right angles; wherefore, the $\triangle$s BEC, ACD, are equiangular, and (E. iv. 6.) $BE : BC :: DC : DA$.

Again (*constr.* and E. xii. 2.)

$$AC^2 = AB^2 + BC^2 + 2\,AB \times BE;$$

and (E. xlvii. 1.)

$$AD^2 = AC^2 + CD^2 = AB^2 + BC^2 + CD^2 + 2AB \times BE;$$

$$\therefore\ AD^2 - (AB^2 + BC^2 + CD^2) = 2\,AB \times BE;$$

also (E. i. 6.) $AB \times BE : AB \times BC :: BE : BC$

$:: DC : DA;$

$$\therefore\ 2AB \times BE : AB \times BC :: 2DC : DA;$$

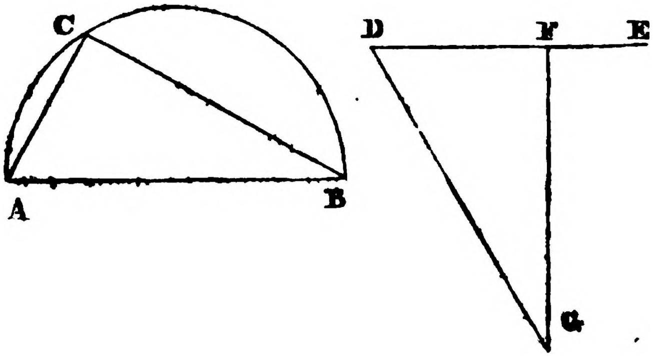
that is,

$$AD^2-(AB^2+BC^2+CD^2) : AB\times BC :: 2DC : DA.$$

PROP. XXIII.

34. PROBLEM. *To divide the semi-circumference of a given circle into two arches, so that the square of the diameter, together with the square of the chord of one of them, shall be in a given ratio to the square of the chord of the other.*

Let $\overset{\frown}{ACB}$ be the given semi-circumference, and



AB the diameter, and let the given ratio be that of DE to DF: It is required to divide $\overset{\frown}{ACB}$ in C, so that $\overline{AB}^2+\overline{AC}^2 : \overline{CB}^2 :: DE : DF$.

Find (Supp. xxi. 6.) the side FG of a square, such that $EF : FD :: 2\,\overline{DF}^2 : \overline{FG}^2$, and let FG be placed at right angles to DF; join D, G; at the point B, in AB, make (E. xxiii. 1.) the $\angle ABC$ equal to the $\angle DGF$: the point C, in which BC meets $\overset{\frown}{ACB}$, is the point of division which was to be found.

For join A, C; and since (E. xxxi. 3.) the $\angle ACB$ is a right angle, as is the $\angle DFG$, and that (*constr.*) the $\angle ABC = \angle DGF$, the two $\triangle$s ACB, DFG, are equiangular.

But (*constr.*) $EF : FD :: 2DF^2 : GF^2$,
$\therefore$ (*componendo*) $DE : FD :: 2DF^2 + GF^2 : GF^2$;
that is, (E. xlvii. 1.) $DE : FD :: DG^2 + DF^2 : GF^2$.

And since the $\triangle$s ACB, DFG, are similar, it is easily shewn, that

$$AB^2 + AC^2 : CB^2 :: DG^2 + DF^2 : GF^2;$$

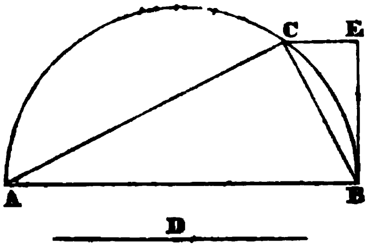
therefore (E. xi. 5.)

$$AB^2 + AC^2 : CB^2 :: DE : DF.$$

PROP. XXIV.

35. **PROBLEM.** *To divide the semi-circumference of a given circle into two arches, so that the square of the diameter shall be in a*[*] *given ratio to the rectangle contained by the chords of those arches.*

Let ACB be the given semi-circumference, and



let the given ratio be that of R to S: It is required to divide $\overset{\frown}{ACB}$ in C, so that

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[*] The given ratio cannot be greater than that of two to one.

$$\overline{AB}^2 : \overline{AC} \times \overline{CB} :: R : S.$$

Find (Supp. xxi. 6.) the side D of a square, such that $R : S :: AB^2 : D^2$; find, also, (E. xi. 6.) a third proportional, BE, to AB and D, and place BE at right angles to AB; through E draw EC parallel to BA, and meeting ACB in C: C is the point which was to be found.

For, join A, C, and B, C; then

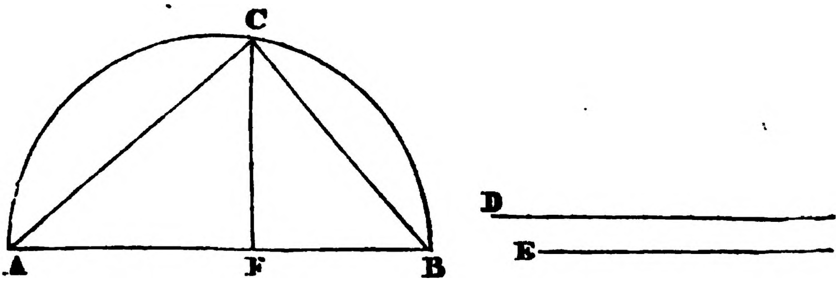
$$AC \times CB = 2 \triangle ACB = AB \times BE$$
$$= (\textit{constr.}\ \text{E. xvii. 6.})\ D^2;$$

therefore (*constr.*) $AB^2 : AC \times CB :: R : S$.

PROP. XXV.

36. PROBLEM. *To divide the semi-circumference of a given circle into two arches, so that the square of the chord of one of them shall be in a given ratio to the rectangle contained by the chord of the other, and the perpendicular drawn to the diameter from the point of division.*

Let ACB be the given semi-circumference; and



let the given ratio be that of AB to D: It is required to divide $\overset{\frown}{ACB}$ in C, so that

$$\overline{CB}^2 : \overline{AC}\times\overline{CF} :: AB : D,$$

CF being drawn from C perpendicular to AB.

Find (E. xi. 6.) a third proportional E, to AB and D; divide (Supp. xlii. 6.) AB in F, so that

$$\overline{AB}\times\overline{BF} : \overline{AF}^2 :: AB : E;$$

from F draw FC at right angles to AB, meeting $\overset{\frown}{ACB}$ in C: C is the point which was to be found.

For, join A, C, and B, C: then, (E. viii. 6. and E. xvii. 6.)

$$CB^2 = AB \times BF;$$

therefore (*constr.*) $CB^2 : AF^2 :: AB : E :: AB^2 : D^2$;

therefore (Supp. i. 6. *Cor.* 2.)

$$CB : AF :: AB : D;$$

and (E. i. 6.) $CB^2 : CB \times AF :: AB : D$;

but (E. viii. 6. and E. xvi. 6.)

$$CB \times AF = AC \times CF;$$

$$\therefore\ CB^2 : AC \times CF :: AB : D.$$

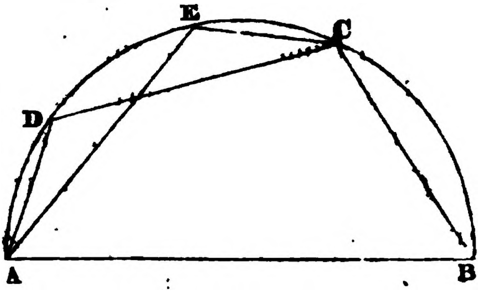
PROP. XXVI.

37. PROBLEM. *To divide the semi-circumference of a given circle into three arches, so that the diameter, and the chords of the three arches, shall be proportionals, according to a given ratio.*

Let $\overset{\frown}{ACB}$ be the given semi-circumference, and let the given ratio be that of R to S: It is required to divide $\overset{\frown}{ACB}$ in C, and D, so that

$$\overline{AB} : \overline{BC} :: \overline{CD} : \overline{DA} :: R : S.$$

Find, and place in the given circle, a fourth proportional BC to R, S and AB; also (Supp. xxx. 6.) divide the arch $\overset{\frown}{AC}$ in D, so that



$$CD : DA :: R : S;$$

and it is manifest (E. xi. 5.) that

$$AB : BC :: CD : DA :: R : S.$$

38. Cor. In like manner, if BC be given, and if $\overset{\frown}{AC}$ be divided in E, so that

$$AB : BC :: AE : EC,$$

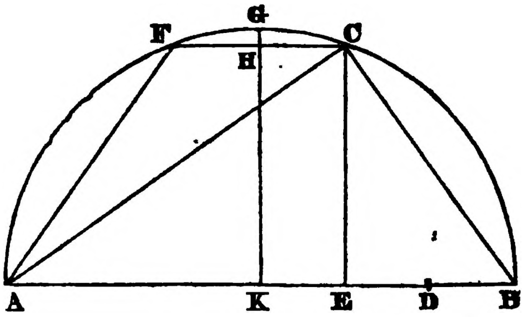
a trapezium $AECB$, may be inscribed in the semicircle ACB, so that $AB \times \overline{EC} = \overline{AE} \times BC$.

Prop. XXVII.

39. Problem. *In a given semi-circle, to inscribe a trapezium, two of the opposite sides of which shall be equal to each other, and the square of either of them equal to the rectangle contained by the diameter and the side opposite to it.*

Let ACB be the given semi-circle, of which AB is the diameter, and K the centre: It is required to inscribe, &c.

Trisect (Supp. xxxiii. 1.) KB in D and E; draw



EC at right angles to AB, meeting $\overset{\frown}{CB}$ in C; from C draw CF parallel to BA, meeting $\overset{\frown}{ACB}$ in F; and join B, C, and A, F; $ABCF$ is the figure, which was to be inscribed in the semi-circle ACB.

Join A, C; and from K draw the semi-diameter KG perpendicular to FC, and therefore (E. iii. 3.) bisecting it in H. Then (Supp. xlvi. 3.) $FA = CB$; also (*constr.*)

$$CF = 2\,CH = 2\,KE = EB;$$

but (*constr.* and E. viii. 6.)

$$AB : BC :: BC : BE \text{ or } CF;$$

therefore (E. xvii. 6.) $\overline{AB} \times \overline{CF} = \overline{CB}^2$.

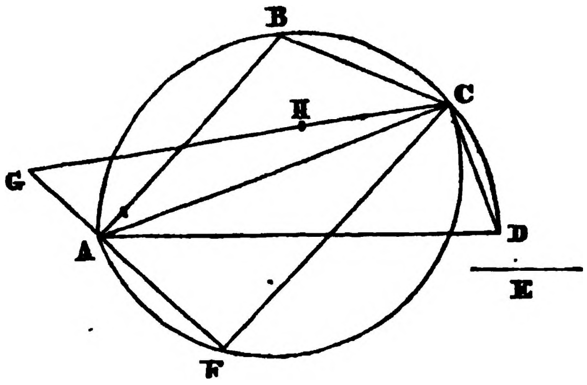
40. Cor. It is manifest (Supp. lvii. 6.) that

$$AB \times \overline{FC} = \tfrac{1}{2}\,\overline{AC}^2.$$

Prop. XXVIII.

41. Problem. *To divide the semi-circumference of a given circle into three arches, so that the rectangle, contained by the chords of the first and second of them, shall be equal to the square of the chord of the third.*

Let *ABCD* be the given semi-circumference: It



is required to divide it in *B* and *C*, so that

$$\overline{AB} \times \overline{BC} = \overline{CD}^2.$$

Assume $\overset{\frown}{DC}$ any arch that is less than a third of the semi-circumference; join *C*, *D*, and *C*, *A*; find (Supp. xxvi. 6.) the side, *E*, of a square, such that

$$AD : 2\overline{DC} :: DC^2 : E^2;$$

upon $\overline{AC}$, as a diameter, describe the circle *AFC*, and in it place *AF* equal to *E*; produce *FA* to *G*, so that $\overline{FG}^2 = 2DC^2$; join *G*, *C*; divide (Supp. Part II. xii. 2.) *GC* in *H*, so that

$$\overline{GH}^2 + \overline{HC}^2 = \overline{CF}^2;$$

in the circle *ABCD*, place *AB* equal to *GH*, and join *B*, *C*: then $AB \times BC = CD^2$.

Join *F*, *C*; then (*constr.* and Supp. Part II. xxii. 3.)

$$\overline{AC}^2 = AB^2 + BC^2 + AF^2 = (\text{E. xlvii. 1.})\ AF^2 + \overline{FC}^2;$$

$\therefore FC^2 = AB^2 + BC^2 =$ (*constr.*) $GH^2 + HC^2$;

and (*constr.*) $AB = GH$; $\therefore AB^2 = GH^2$;

$$BC^2 = HC^2, \text{ and } BC = HC.$$

Again, (E. iv. 2.)

$GC^2 = GH^2 + HC^2 + 2\,GH \times HC = FC^2 + 2\,AB \times BC$;

also, (E. xlvii. 1.)

$GC^2 = CF^2 + FG^2 =$ (*constr.*) $CF^2 + 2\,CD^2$;

$\therefore\ 2\,AB \times BC = 2\,CD^2$; and $AB \times BC = CD^2$.

A SUPPLEMENT

TO THE

ELEMENTS OF EUCLID.

PART II.

BOOK IV.

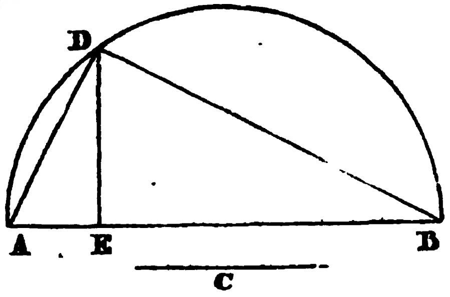
PROP. I.

1. **PROBLEM.** *THE hypotenuse of a right-angled triangle, and the ratio of one of the two remaining sides to the perpendicular drawn to the hypotenuse from the right angle being given, to construct the triangle.*

Let AB be the hypotenuse, and let the given ratio of one of the remaining sides to the perpendicular, be that of AB to C: It is required to construct the triangle.

Upon AB, as a diameter, describe the circle ADB; in it place AD equal to C, and join D, B: then is ADB the triangle which was to be constructed.

For draw *DE* perpendicular to *AB*: the ∠ *ADB*



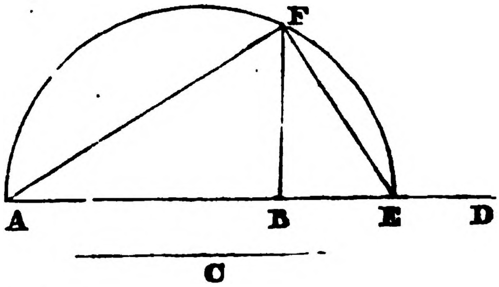
is (E. xxxi. 3.) a right angle, and (E. viii. 6.)

AB : *AD* or *C* :: *DB* : *DE*.

Prop. II.

2. **Problem.** *One of the two sides about the right angle of a right-angled triangle being given, and a mean proportional between the other side about the right angle, and the aggregate of that other side and the hypotenuse, to construct the triangle.*

Let *AB* be the given side of the right-angled triangle,



and *C* the given mean proportional between the other side and the aggregate of that other side and the hypotenuse: It is required to construct the triangle.

From *AB* produced, cut off *BD*, equal to a third proportional (E. xi. 6.) to *AB* and *C*; divide (Supp. Part II. xvi. 2.) *BD* in *E*, so that $\overline{AE}\times\overline{EB}=\overline{ED}^2$; upon *AE*, as a diameter, describe the circle *AFE*; from *B* draw *BF* at right angles to *AE*, meeting $\overset{\frown}{AFE}$ in *F*; and join *A*, *F* and *E*, *F*: *AFB* is the triangle, which was to be constructed.

For (*constr.* and E. xxxi. 3.) the ∠ *AFE* is a right angle; and (E. viii. 6. and xvii. 6.)

$$\overline{AE}\times\overline{EB}=\overline{EF}^2=(\textit{constr.})\ \overline{ED}^2;$$
$$\therefore EF=ED.$$

Again (*constr.* and E. xvii. 6.) $\overline{AB}\times\overline{BD}=C^2$;
that is, (E. i. 2.) $\overline{AB}\times\overline{BE}+\overline{AB}\times\overline{ED}=C^2$,
therefore (E. viii. 6.)

$$\overline{BF}^2+AB\times\overline{ED}\ \text{or}\ \overline{BF}^2+AB\times\overline{EF}=C^2.$$

But (E, viii. 6. and E. xvi. 6.)

$$\overline{AB}\times\overline{EF}=\overline{AF}\times\overline{FB};$$
$$\therefore\ \overline{BF}^2+\overline{AF}\times\overline{BF}=C^2;$$

or (E. i. 2.)

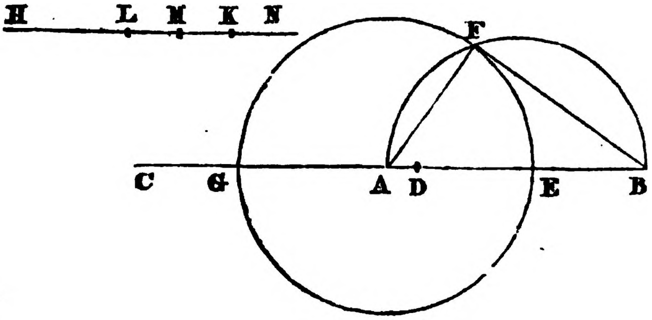
$$BF\times(AF+FB)=C^2:$$

Wherefore (E. v. 76.) the right-angled △ *AFC* is such that *C* is a mean proportional between the side *FB*, and the aggregate *AF*+*FB* of that side, and the hypotenuse; and it has the line *AB* for its remaining side.

PROP. III.

3. PROBLEM. *The hypotenuse of a right-angled triangle, and the ratio which the aggregate of the hypotenuse, and one of the two remaining sides has to the third side, being given, to construct the triangle.*

Let *AB* be the hypotenuse, and let the ratio be



that of *KH* to *KL*: It is required, &c.

From *KH* cut off *KM* equal to a third proportional to *KH* and *KL*; produce *MK* to *N*, so that *MK*=*KN*; produce, also, *BA* to *C*, so that *AC*= *BA*; find (E. xii. 1.) a fourth proportional to *HN*, *NM*, and *CB*, and make *BD* equal to it; bisect *DB* in *E*; upon *AB* as a diameter, describe the circle *AFB*, and in it place $\overline{AF}$ equal to *AE*; join *F*, *B*: the △ *AFB* is that, which was to be constructed.

For (*constr.* and E. xxxi. 3.) the △ *AFB* is right-angled at *F*, and if from the centre *A*, at the distance *AE*, a circle *EFG* be described, *BF* will touch it in *F*.

(*constr.*) $HN : NM :: BC : BD$,

therefore (*dividendo*)

$$HM : MN :: CD \text{ or } BA+AD : BD;$$

and *MK* is the half of *MN*, and *DE* the half of *BD*;

$$\therefore HM : MK :: BA+AD : DE;$$

and (*componendo*)

$$HK : KM :: AB+AE : DE,$$

that is, (*constr.*)

$$HK : KM :: AB+AF : DE;$$

but (*constr.*) $HK : KL :: KL : KM$,
and (E. xxxvi. 3. and E. xvii. 6.)

BG or $AB+AF : BF :: BF : BE$ or DE;
therefore (E. xi. 5.)

$$HK^2 : KL^2 :: (AB+AF)^2 : BF^2,$$

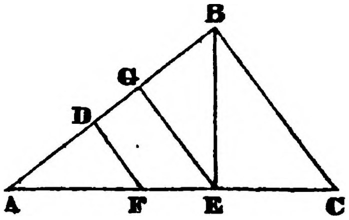
and (Supp. i. 6. *Cor.* 2.)

$$HK : KL :: AB+AF : BF.$$

PROP. IV.

4. THEOREM. *If the three sides of a right-angled triangle be continual proportionals, and if the greater of the two sides, about the right angle, be cut in extreme and mean ratio, its greater segment shall be equal to the perpendicular drawn to the hypotenuse from the right angle.*

Let the three sides of the $\triangle ABC$, right-angled at B,



be continual proportionals; let AB, which is greater than BC, be cut in extreme and mean ratio in D, and let BE be perpendicular to AC : $BD=BE$.

Draw DF parallel to BC, and (E. ii. 6.)

$$CF : FA :: BD : DA;$$

wherefore CA is cut in extreme and mean ratio in F, as (Supp. lxxiii. 6. *Cor.* 2.) it is, also, in E;

$$\therefore AE=CF:$$

Also (E. viii. 6.)

$$AB : BE :: AC : CB \text{ or } AE \text{ or } FC;$$

and (E. ii. 6.) $AC : FC :: AB : BD$;
therefore (E. xi. 5.) $AB : BE :: AB : BD$;
therefore (E. ix. 5.) $BE = BD$.

5. Cor. If EG be drawn perpendicular to AB, $EG = EC$.

For (E. iv. 6.) $AC : CB$ or $AE :: AE : EG$; and (Supp. lxxiii. 6. *Cor.* 2.)

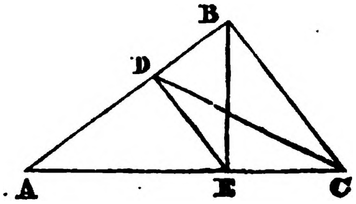
$$AC : AE :: AE : EC;$$

therefore (E. xi. 5. and E. ix. 5.) $EG = EC$.

Prop. V.

6. Theorem. *If the three sides of a right-angled triangle be continual proportionals, the straight line, which bisects the greater of the two acute angles, shall cut the opposite side in extreme and mean ratio.*

Let the three sides of the △ ABC, right-angled at B, be continual proportionals; let AB be greater than BC, and, therefore, the ∠ BCA greater than the ∠ BAC; and let CD bisect the ∠ BCA, and cut AB in D:



$$BA : AD :: AD : DB.$$

For, draw BE perpendicular to AC, and join D, E. Then (Supp. lxxiii. 6. *Cor.* 2.) AC is divided in extreme and mean ratio, in E, and $AE = BC$.

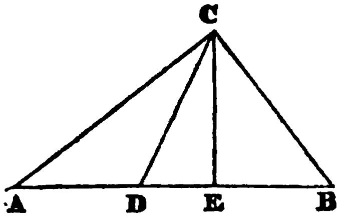
Also (E. iii. 6.) $AC : CB$ or $AE :: AD : DB$; wherefore (E. ii. 6.) DE is parallel to BC,

and (E. iv. 6.) $CA : AE :: BA : AD$;
therefore (E. xi. 5.) $BA : AD :: AD : DB$;
wherefore AB is cut in extreme and mean ratio in D.

PROP. VI.

7. PROBLEM. *To make an isosceles triangle, such that a perpendicular drawn to either of the equal sides from the opposite angle, shall cut that side in extreme and mean ratio.*

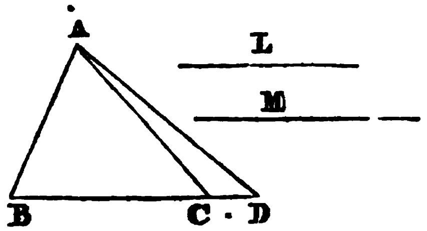
Assume any finite straight line AB; make (Supp. lxxiii. 6.) AB the hypotenuse of a right-angled △ ACB, having its three sides continual proportionals; from BA cut off BD equal to BC the least side, and join D, C: the △ BDC is that which was to be constructed.



For draw CE perpendicular to BD; then $AE = CB$; and (*constr.*) $BC = BD$; $\therefore AE = BD$, and $AD = BE$; but (Supp. lxxiii. 6.) AB is cut in extreme and mean ratio in E, and (Supp. lxxii. 6.) DB is, therefore, cut in extreme and mean ratio in E.

PROP. VII.

8. PROBLEM. *From the summit of a given triangle, to draw a straight line to meet the base produced, so that the rectangle contained by that line and one of the two sides shall be in a given ratio to the square of the other side.*



Let *ABC* be the triangle, of which *A* is the summit, and *BC* the base, and let the ratio be that of *L* to *BC*: It is required to draw $\overline{AD}$ meeting *BC* produced in *D*, so that

$$AD \times \overline{AB} : \overline{AC}^2 :: L : BC.$$

Find (E. xi. 6.) a third proportional *M*, to *AB* and *BC*; find, also, (E. xii. 6.) a fourth proportional to *BC*, *L* and *M*, and make *BCD* equal to it; join *A*, *D*: *AD* is the line which was to be drawn.

For (*constr*. and E. xvii. 6.) $AC^2 = M \times AB$,
and (E. i. 6.) $BA \times AD : M \times AB :: AD : M$;
but (*constr.*) $AD : M :: L : BC$;
therefore (E. xi. 6.)

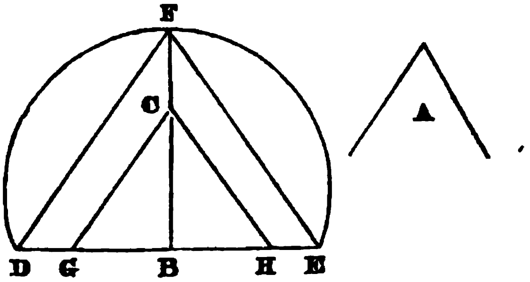
$$AD \times AB : M \times AB \text{ or } AC^2 :: L : BC.$$

PROP. VIII.

9. **PROBLEM.** *The angle at the summit of a triangle, the perpendicular drawn from it to the base, and the ratio of the segments into which the perpendicular divides the base being given, to construct the triangle.*

Let *A* be the angle at the summit, *CB* the perpendicular, and let the ratio of the segments of the base be that of *DB* to *BE*, *DBE* being placed at right angles to *CB*: It is required to construct the triangle.

Upon *DE* (E. xxxiii. 3.) describe a segment of a circle *DFE*, containing an angle equal to the ∠*A*, and let *BC*, produced if necessary, meet $\widehat{DFE}$ in *F*; join *F*, *D*, and *F*, *E*; and from *C* draw *CE* parallel to *FD*, and *CH* parallel to *FE*, meeting *DE*, produced if necessary in *G* and *H* respectively: the △ *GCH* is that which was to be constructed.



For it has *BC* for the perpendicular; also (*constr.* and E. xxix. 1.) the ∠ *GCH* = ∠ *DFE* = (*constr.*) ∠ *A*: And (*constr.* and E. iv. 6.)

$$BG : BD :: BC : BF :: BH : BE;$$

therefore (*permutando*)

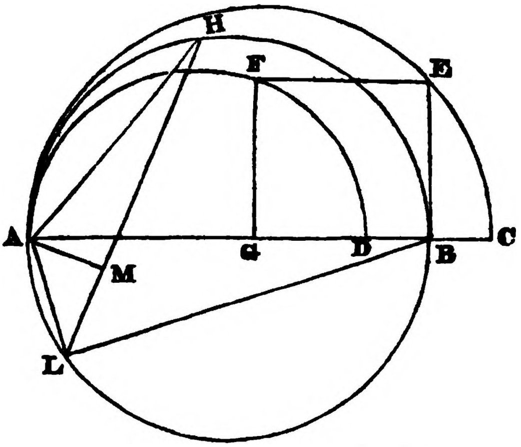
$$BG : BH :: BD : BE.$$

Prop. IX.

10. Problem. *The diameter of a circle described about a triangle, the aggregate of the two sides, and the altitude of the triangle being given, to construct it.*

Let *AB* be the diameter, *BC* the altitude, and *AD* the aggregate of the two sides: It is required to construct the triangle.

Upon *AC*, *AB*, and *AD* as diameters, describe the circles *AEC, AHBL, AFD*; from *B*, draw *BE*



at right angles to *AC* meeting $\overset{\frown}{AEC}$ in *E*; draw *EF* parallel to *AB*, meeting $\overset{\frown}{AFB}$ in *F*, draw *FG* perpendicular to *AD*; and in the circle *AHBL* place *AH* equal to *AG*, and *AL* equal to *GD*, and join *H, L*: the △ *AHL* is that, which was to be constructed.

For draw *AM* perpendicular to *HL*, and join *L, B*. Since (*constr.* and E. xxxi. 3.) the two △s *BLA, HMA*, are right-angled, and (E. xxi. 3.) have, also, the ∠s *AHM, ABL*, equal to one another, they are equiangular;

therefore (E. iv. 6.) $AB : AL :: AH : AM$.

But (*constr.* and E. xxxiv. 1.) $\overline{EB}^2=\overline{FG}^2$;

that is, (E. viii. 6.)

$\overline{AB}\times\overline{BC}=\overline{AG}\times\overline{GD}=$ (*constr.*) $\overline{AH}\times\overline{AL}$;

therefore (E. xvi. 6.) $AB : AL :: AH : BC$;

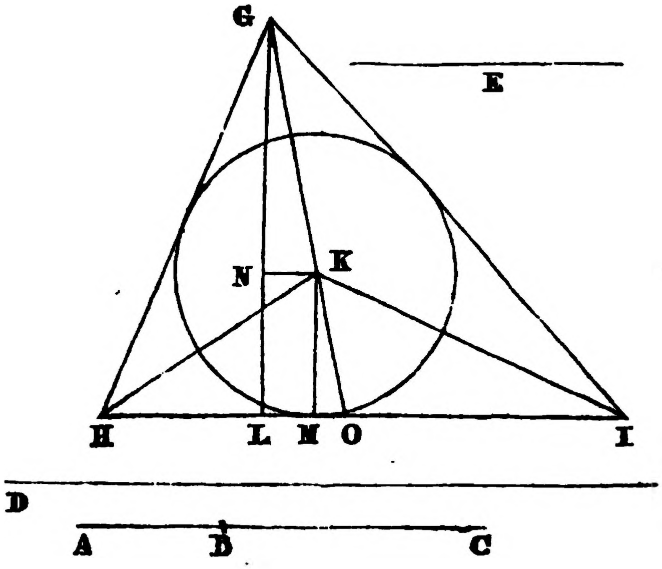
therefore (E. ix. 5.) $AM=BC$:

and *BC* is the altitude; and $AH+AL=AD$, the aggregate of the two sides.

PROP. X.

11. **PROBLEM.** *The altitude of a triangle, the rectangle contained by the base, and the aggregate of the two remaining sides, and the diameter of a circle inscribed in the triangle, being given, to construct it.*

Let AB be the half of the given diameter, AC the given altitude, and D the side of a square equal to the given rectangle contained by the base, and the aggregate of the two remaining sides: It is required to construct the triangle.



Find (Supp. xxi. 6.) a square E^2 such that

$$CB : BA :: D^2 : E^2;$$

find, also, a third proportional F to E and D; and construct (Supp. xcviii. 3.) the $\triangle$ GHI, having its altitude GL equal to AC, its base HI equal to E, and the aggregate of the sides $GH+GI$, equal to F: the $\triangle$ GHI is that, which was to be constructed.

Find (E. iv. 4.) the centre K, of the circle inscribed in the $\triangle$ GHI, by bisecting the $\angle$s GHI, GIH; from K, draw KM perpendicular to HI, and KN perpendicular to GL; also, join G, K, and produce GK to meet HI in O.

Then (*constr.* and E. xvii. 6.) $E \times F = D^2$;
that is, (*constr.*) $\overline{HI}\,(GH+GI) = D^2$;
therefore (*constr.*)
$HI\,(GH+HI) : HI^2 :: CB : BA$;
that is, (E. i. 6.)
$GH+GI : HI :: CB : BA$.
Again (*constr.* and E. iii. 6.)
$GI : GH :: IO : HO$,
(*componendo*)
$GI+GH : GH :: HI : HO$,
(*permutando*)
$GI+GH : HI :: GH : HO :: GK : KO$;
therefore (*constr.* and E. ii. 6.)
$GI+GH : HI :: GN : NL$;
that is, (*constr.*)
$F : E :: GN : NL$,
or (*constr.*)
$CB : BA :: GN : NL$;
therefore (*componendo*)
$CA : AB :: GL : NL$ or KM;

but (*constr.*) $GL=CA$; therefore (E. ix. 5.) $KM = AB$; wherefore GHI is the triangle, which was to be constructed.

12. Cor. It has been shewn that

$$GI+GH : HI :: GN : NL;$$

therefore (*componendo*)

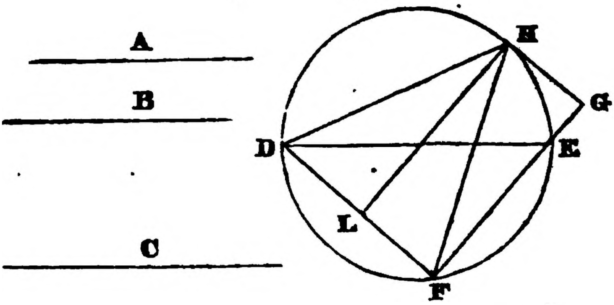
$$GI+GH+HI : HI :: GL : NL \text{ or } KM;$$

that is, the aggregate of the three sides of a triangle is to the base as the perpendicular is to the semidiameter of the inscribed circle.

Prop. XI.

13. Problem. *The base of a triangle, the perpendicular drawn to it from the opposite angle, and a square equal to the rectangle contained by the two remaining sides, being given, to construct the triangle.*

Let A be the base, B the perpendicular, and let C be the side of a square equal to the rectangle contained by the two remaining sides of the triangle: It is required to construct it.



Find (E. xi. 6.) a third proportional DE to B and C; upon DE, as a diameter, describe the circle DEF; in it place DF, equal to A; from F draw FG, at right angles to DF, and make it equal to B; from G draw GH parallel to FD, meeting $\overset{\frown}{DHF}$ in H, and join D, H, and H, F: the $\triangle\ HDF$ is that which was to be constructed.

For draw HL parallel to GF, and therefore perpendicular to DF, and (E. xxxiv. 1.)

$$HL = GF = B;$$

also (*constr.*) $DF = A$; and (Supp. lvi. 6.)

$$\overline{HD} \times \overline{HF} = \overline{HL} \times \overline{DE} = (\textit{constr.})\ C^2.$$

PROP. XII.

14. THEOREM. *The diameter of a trapezium, which has two parallel sides, divides it into two triangles, which are to one another as those parallel sides.*

For the two triangles have equal altitudes, and, therefore, (E. i. 6.) are to one another as their bases.

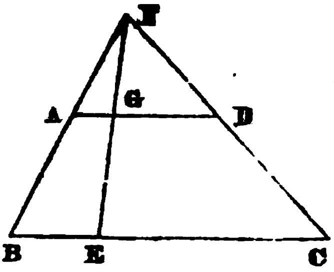
15. COR. Trapeziums, which are between the same parallels, are to one another as the aggregates of their parallel sides.

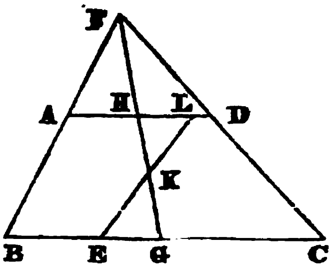
PROP. XIII.

16. PROBLEM. *To divide a given trapezium, having two parallel sides, by a straight line passing through a given point of one of them, into two parts, which shall have to one another a given ratio.*

Let $ABCD$ be the trapezium, having the sides AD, BC, parallel to each other; let E be a point in one of the two parallel sides, as BC; and let the ratio be that of R to S: It is required to divide $ABCD$, by a straight line passing through E, into two parts, which shall be to each other in the ratio of R to S.

Produce BA, CD, until they meet in F; and,





first, if BE be to EC, as R is to S, join E, F, and EF is the line which was to be drawn.

For (E. iv. 6.)

$$AG : BE :: GF : EF :: GD : EC;$$

therefore (*componendo*)

$$AG+BE : BE :: GD+EC : EC,$$

and (*permutando*)

$$AG+BE : GD+EC :: BE : EC;$$

But (Supp. Part II. xii. 4. *Cor.*)

$$ABEG : GECD :: AG+BE : GD+EC;$$

therefore (E. xi. 5.)

$$ABEG : GECD :: BE : EC.$$

But if BE be not to EC as R is to S, divide (E. x. 6.) BC in G, so that

$$BG : GC :: R : S;$$

draw $\overline{FG}$, and let it cut AD in H; bisect HG in K; join E, K, and produce EK to meet AD in L; EL is the line which was to be drawn.

For (*constr.* E. xxix. 1. E. xxvi. 1.) the $\triangle\, HKL = \triangle\, EKG$; add to each the figure $AHKEB$, and it is manifest, that $ALEB = AHGB$; in the same manner it may be shewn, that $DLEC = DHGC$; and, by the former case,

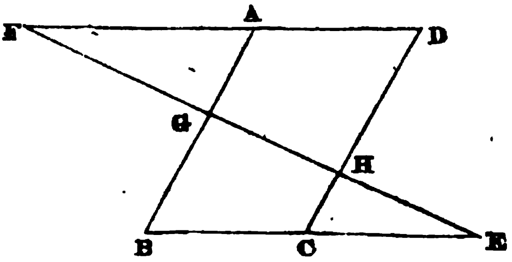
$$AHGC : DHGC :: BG : GC;$$

$$\therefore\ ALEB : DLEC :: BG : GC :: R : S.$$

PROP. XIV.

17. PROBLEM. *From a given point, in the base produced, of a given parallelogram, to draw a straight line, meeting the opposite side produced, so as to cut off, from the parallelogram a trapezium, which shall be in a given ratio to the triangle contained by the segment of the opposite side produced, that of the side adjacent to it, and that of the cutting line.*

Let $ABCD$ be the parallelogram, E the point in BC produced, and let the ratio be that of BC to K: It is required to draw, from E, $\overline{EF}$, meeting DA produced in F, so that



$$DAGH : \triangle FAG :: BC : K.$$

Produce (Supp. Part II. vii. 2.) DA to F, so that

$$\overline{DA}^2 + 2DA \times \overline{AF} : AF^2 :: BC : K;$$

join E, F; EF is the straight line which was to be drawn.

For the $\triangle$s DFH, AFG, are similar, therefore (E. xxix. 6.)

$$\triangle DFH : \triangle AFG :: DF^2 : AF^2;$$

therefore (*dividendo*) trapezium

$$DAGH : \triangle AFG :: \overline{DF}^2 - AF^2 : AF^2;$$

that is, (E. vii. 2.) trapezium

$$DAGH : \triangle AFG :: DA^2 + 2DA \times AF : AF^2;$$

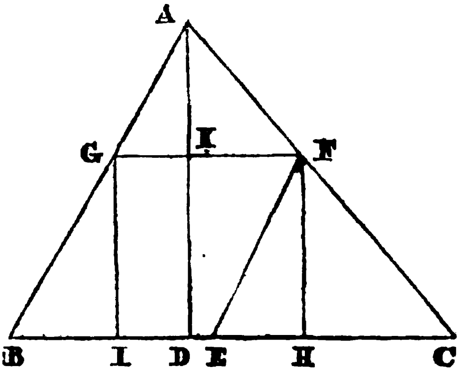
therefore (*constr.* and E. xi. 5.)

$$DAGH : \triangle AFG :: BC : K.$$

PROP. XV.

18. PROBLEM. *To inscribe a square in a given triangle.*

Let ABC be the triangle: It is required to inscribe in it a square.



Draw AD perpendicular to BC. From BC cut off BE, equal to a fourth proportional to $BC + AD$, BC and AD; and through E, draw EF parallel to BA, meeting AC in F; from F draw FG parallel to BC; and also from F and G, draw FH, GI, perpendicular to BC: $FGIH$ is the square which was to be inscribed.

For $BEFG$ is, by construction, a parallelogram; therefore, (E. xxxiv. i.) $FG = BE$.

Let FG cut AD in K. Then (*constr.*)

$$BC+AD : BC :: AD : BE;$$

therefore (*dividendo*)

$$AD : BC :: AD-BE : BE,$$

and (*permutando*)

$$AD : AD-BE :: BC : BE,$$

but (E. iv. 6.)

$$BC : GF \text{ or } BE :: AC : AF :: AD : AK;$$

therefore (E. xi. 5.)

$$AD : AD-BE :: AD : AK;$$

$$\therefore AD-BE=AK=AD-KD;$$

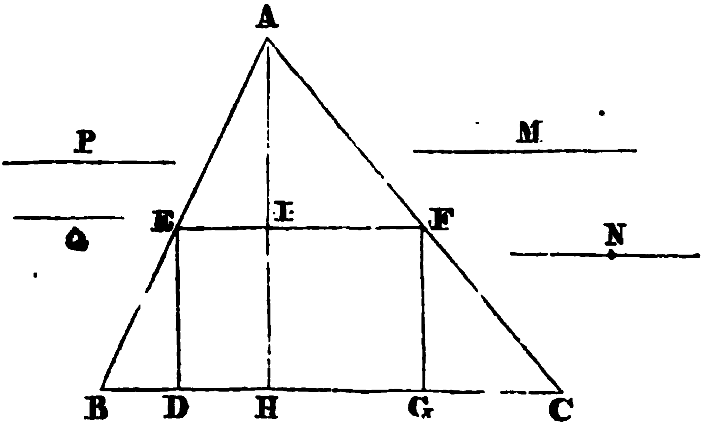
$$\therefore BE \text{ or } GF=KD=(\text{E. xxxiv. 1.})\ GI \text{ or } FH.$$

Wherefore, the parallelogram $FGIH$ is equilateral, and (*constr.* and E. xxxiv. 1.) its angles are right angles. Therefore it is a square.

PROP. XVI.

19. PROBLEM. *In a given triangle, to inscribe a rectangle, the sides of which shall be to each other in a given ratio.*

Let ABC be the triangle, and let the ratio be that



of M to N: It is required to inscribe in the △ ABC a rectangle $DEFG$, such that

$$EF : FG :: M : N.$$

From A draw AH perpendicular to BC; find P such, that

$$BC : AH :: M : P;$$

and Q such, that

$$N+P : M :: AH : Q;$$

and from AC cut off AF equal to a fourth proportional to BC, Q, and AC; from F draw FE parallel to CB, cutting AH in I, and, from E and F, draw ED and FG perpendicular to BC. The rectangle DF is that, which was to be inscribed.

For (*constr.*) $BC : Q :: AC : AF$,
and (E. iv. 6.) $AC : AF :: BC : EF$;
therefore (E. xi. 5. and E. ix. 5.) $EF = Q$.

Again, (*constr.* and E. xvi. 6.)

$$\overline{BC} \times P = \overline{AH} \times M, \text{ and } (N+P) \times Q = \overline{AH} \times M;$$

$$\therefore \overset{\frown}{BC} \times P = (N+P) \times Q = (N+P) \times EF;$$

therefore (E. xvi. 6.)

$$N+P : P :: BC : CF;$$

but (E. iv. 6.)

$$BC : EF :: AC : AF :: AH : AI;$$

therefore (E. xi. 5.)

$$N+P : P :: AH : AI,$$

and (*dividendo*)

$$N : P :: IH \text{ or } FG : AI.$$

But, since

$$BC : EF :: AH : AI,$$

(*permutando*)

$$BC : AH :: EF : AI,$$

And (*constr.*) $BC : AH :: M : P$,

$\therefore EF : AI :: M : P$,

and, (as was shewn)

$AI : FG :: P : N$;

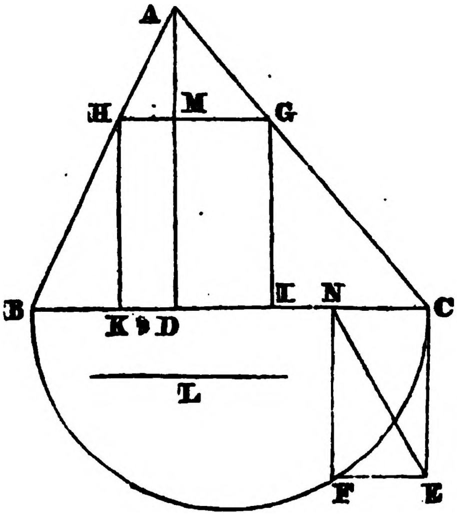
therefore (*ex æquo*)

$EF : FG :: M : N$.

Prop. XVII.

20. **Problem.** *In a given triangle, to inscribe a rectangle, which shall be equal to a given square.*

Let ABC be the triangle, and L the side of the square: It is required to inscribe, in ABC a rectangle equal to L^2.



From A draw AD perpendicular to BC; find (Supp. xxi. 6.) a square which is to L^2, as BC is to AD; let CE be its side, and place CE at right angles to BC; upon BC, as a diameter, describe the circle BFC: through E draw EF parallel to CB[*], meeting $\overset{\frown}{BFC}$ in F; from AC cut off AG, equal to a fourth proportional to BC, EF, and AC; from G draw GH parallel to CB, and from G and H draw GI, and HK, perpendicular to BC: the inscribed rectangle $GK = L^2$.

For let GH cut AD in M, and draw FN parallel to EC.

Then since (*constr.*)

$CA : AG :: BC : EF$ or NC,

and (E. iv. 6.)

$CA : AG :: BC : HG$;

therefore (E. xi. 5. E. ix. 5.) $HG = NC$.

Again, (E. iv, 6,)

$CA : AG :: AD : AM$,

and $CA : AG :: BC : HG$;

$\therefore BC : HG :: AD : AM$,

and (*permutando*)

$BC : AD :: HG : AM$;

that is,

$BC : AD :: NC : AM$;

therefore (E. xix. 5.)

$BN : MD$ or $GI :: BC : AD$;

but (E. i. 6.)

$BN : GI :: \overline{BN} \times \overline{NC} : \overline{GI} \times \overline{GH}$;

that is,

$BN : GI :: \overline{NF}^2$ or $\overline{CE}^2 : \overline{GI} \times \overline{GH}$;

---

[*] If EF do not meet the circumference of AFC, the problem is impossible. The inscribed rectangle cannot exceed that which has for its altitude the half of the altitude of the triangle.

therefore (E. xi. 5.)

$$BC : AD :: \overline{CE}^2 : \overline{GI} \times \overline{GH}.$$

But (*constr.*)

$$BC : AD :: \overline{CE}^2 : L^2,$$

$$\therefore \overline{GI} \times \overline{GH} = L^2.$$

PROP. XVIII.

21. PROBLEM. *In a given triangle, to inscribe a rectangle, which shall be to the triangle in a given ratio.*

Find (Supp. xxi. 6.) a square which shall be to the triangle in the given ratio; then (Supp. Part II. xvii. 4.) inscribe, in the triangle, a rectangle equal to that square; and (E. vii. 5.) it will be to the triangle in the given ratio.

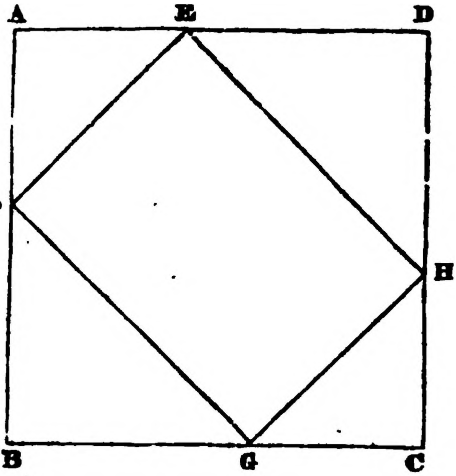
PROP. XIX.

22. PROBLEM. *In a given square to inscribe an oblong, the two adjacent sides of which shall be to one another in a given ratio.*

Let $ABCD$ be the square, and let the ratio be that of AE to ED: It is required to inscribe in $ABCD$ an oblong, the adjacent sides of which shall be to each other as AE is to ED.

From AB, CB, and CD, cut off AF, CG, CH, each equal to AE, and join E, F and F, G, and G, H and H, E: $EFGH$ is the oblong which was to be inscribed.

For (*constr*. E. v. 1. E. xxxii. 1.) the $\angle$s AFE, AEF, DEH, DHE, are each of them the half of a right angle; wherefore, (E. xiii. 1.) the $\angle$ FEH is a right angle; and, in like manner, it may be shewn,



that the ∠s *EFG*, *FGH*, *GHE*, are each of them right angles; therefore, the figure *EFGH* is an oblong.

Again, since the △s *AEF*, *DEH*, are equiangular,

(E. iv. 6.) *FE* : *EA* :: *HE* : *ED*,

therefore (*permutando*) *FE* : *EH* :: *AE* : *ED*.

PROP. XX.

23. PROBLEM. *In a given square, to inscribe an oblong equal to a given rectilineal figure not greater than the half of the square.*

Let *ABCD* (see Fig. to Art. 22.) be the square: It is required to inscribe in it an oblong, equal to a rectilineal figure, not greater than the half of it.

Find (E. xiv. 2.) a square L^2, equal to the rectilineal figure; divide (Supp. lxxi. 3.) *AB* in *F*, so that $\overline{AF} \times \overline{FB}$ shall be equal to the half of L^2; from *BC*, *CD*, *DA*, cut off *BG*, *DH*, *DE*, each equal to

FB; and draw $\overline{EF}, \overline{FG}, \overline{GH}, \overline{HE}$: *EFGH* is the oblong, which was to be inscribed.

For it may be shewn, as in Art. 22. that *EFGH* is a rectangle. Also, the square *ABCD* is made up of the rectangle *EG*, and of the four △s *AEF*, *CHG*, *BFG*, *DEH*; and it is manifest, that

$$\triangle AEF + \triangle CHG = AF^2,$$

$$\text{and } \triangle BFG + \triangle DEH = FB^2;$$

also (E. iv. 2.) $\overline{AB}^2 = AF^2 + FB^2 + 2\,AF \times FB$; therefore the rectangle $EG = 2\,AF \times FB =$ (*constr.*) L^2; and L^2 was made equal to the rectilineal figure.

PROP. XXI.

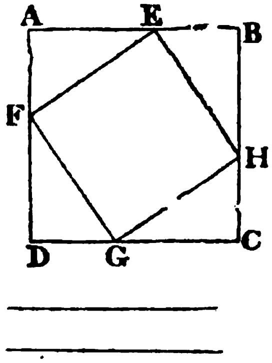
24. **PROBLEM.** *In a given square to inscribe an oblong, which shall be to the square in a given ratio; not greater than that of one to two.*

Find (Supp. xxi. 6.) a square which shall be to the square in the given ratio; inscribe (Supp. Part II. xx. 4.) in the square, an oblong equal to the square so found; and (E. vii. 5.) it will be to the square, in the given ratio.

PROP. XXII.

25. **PROBLEM.** *In a given square, to inscribe a square which shall be to it in a given ratio, not less than that of one to two.*

Let *ABCD* be the square: It is required to inscribe in it a square, which shall be to it in the ratio of *R* to *S*. Find a mean proportional *T*, between *R* and *S*; find, also, a fourth proportional *r*, to *T*, *R*, and AB; divide (Supp. Part II. xxii. 2.) AB in E,



so that $\overline{AE}^2+\overline{EB}^2=r^2$; from AD, DC, CB, cut off AF, DG, CH, each equal to EB, and draw $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, $\overline{HE}$: EG is the square which was to be inscribed.

For (*constr.* and Supp. xxxv. 1.) EG is a square: Also, (*constr.*) and E. xlvii. 1.)

$$EF^2=AE^2=EB^2.\ r^2;$$

and (*constr.*)

$$R\ :\ T\ ::\ r\ :\ AB;$$

therefore (Supp. i. 6. *Cor.* 1.)

$$R^2\ :\ T^2\ ::\ r^2 \text{ or } EG\ :\ AB,$$

but (*constr.*)

$$R^2\ :\ T^2\ ::\ R\ :\ S;$$

therefore (E. xi. 5.)

$$R\ :\ S\ ::\ r^2\ :\ AB^2.$$

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A
SUPPLEMENT
TO THE
ELEMENTS OF EUCLID.

PART II.

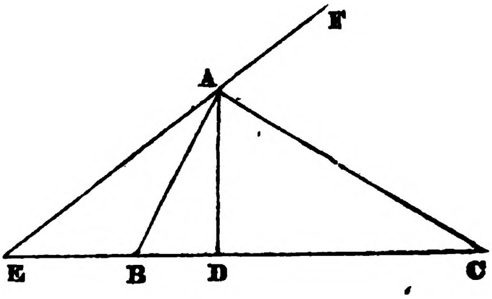
BOOK V.

Prop. I.

1. **Theorem.** *If, from the right angle of a right-angled triangle, two straight lines be drawn, making equal angles with either of the sides, the straight line, which is made up of the hypotenuse and the exterior segment, is cut harmonically by that side, and the straight line so drawn, which is within the triangle.*

Let *ABC* be a right-angled triangle, having the angle *BAC* a right angle; and let *AD* and *AE*, cutting *CB* in *D*, and *CB* produced in *E*, make equal angles with *AB*: *CE* is cut harmonically in *B* and *D*.

Produce *EA* to *F*: then (E. xiii. 1.) the $\angle$s *EAB*, *BAC*, *CAF*, are, together, equal to two right angles, and (*hyp.*) the $\angle BAC$ is a right angle; therefore the



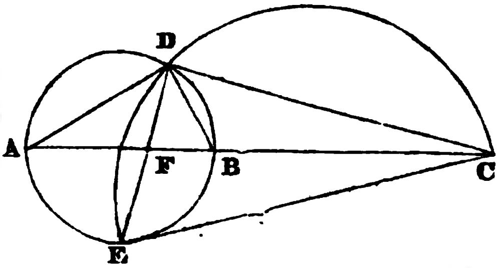
$$\angle EAB + \angle CAF = \angle BAC = \angle BAD + \angle DAC;$$

but (*hyp.*) the $\angle EAB = \angle BAD$, therefore the $\angle CAF = \angle DAC$; therefore (Supp. ii. 6. *Cor.* 1.) *CE* is cut harmonically in *B* and *D*.

Prop. II.

2. Problem. *To find a third harmonic proportional to two given unequal straight lines.*

Let *AC* and *AB* be the two given unequal straight lines: It is required to find a third harmonic proportional to *AC* and *AB*.



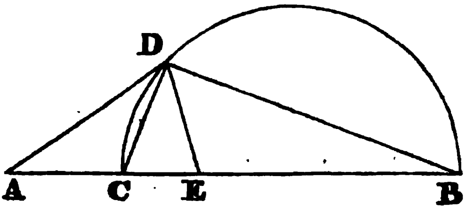
Upon *AB,* as a diameter, describe the circle *ADBE*; bisect $\widehat{AEB}$ in *E*; join *E, C*; upon *EC,* as a diameter, describe the circle *EDC*, and let $\widehat{EDC}$ cut $\widehat{ADB}$ in *D*; join *E, D*, and let *ED* cut *AB* in *F*: *AF* is the line, which was to be found.

For (*constr.* and E. xxvii. 3.) the $\angle ADF = BDF$, and (E. xxxi. 3.) the $\angle FDC$ is a right angle: wherefore (Supp. Part II. i. 5.) *AC, AB* and *AF*, are harmonic proportionals.

Prop. III.

3. Problem. *To find an harmonic mean proportional between two given unequal straight lines.*

Let *AB* and *AC* be the two given unequal straight lines: It is required to find an harmonic mean proportional between them.



Upon *CB,* as a diameter, describe the circle *CDB*; take any point *D* in its circumference, join *D, A,* and *D, C,* and at the point *D,* in *CD,* make (E. xxiii. 1.) the $\angle CDE$ equal to the $\angle ADC$: *AE* is the line, which was to be found.

For, join *D, B*; and since (*constr.* and E. xxxi. 3.) the $\angle CDB$ is a right angle, and the

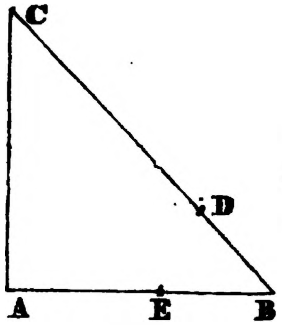
$$\angle ADC = \angle EDC,$$

AE is (Supp. Part II. i. 5.) an harmonic mean between *AB* and *AC.*

PROP. IV.

4. PROBLEM. *To divide a given finite straight line in extreme and mean harmonic ratio.*

Let AB be the given finite straight line: It is



required to divide it in extreme and mean harmonic ratio.

From A, draw AC at right angles to AB, and make AC equal to AB; join C, B; from CB, cut off CD, equal to CA or AB, and from BA, cut off BE equal to BD: AB is divided in extreme and mean harmonic ratio in E.

For (E. iv. 2.) $\overline{DC}^2+\overline{DB}^2+2DC\times DB=\overline{BC}^2$

= (E. xlvii. 1. and *constr.*) $2\overline{AC}^2=2\overline{DC}^2$;

from these equals take the common square DC^2,

and $DB^2+2DC\times DB=DC^2$,

that is, (E. i. 2.) $DB\times(DB+2DC)=DC^2$;

and $2DB\times(DB+2DC)=2DC^2$;

therefore (E. xvii. 6.)

$$DB+2DC:2DC::DC:2DB;$$

therefore (*dividendo*)

$$DB:2DC::DC-2DB:2DB,$$

and $DB:DC::DC-2DB:DB$;

therefore (*invertendo*)

$$DC : DB :: DB : DC-2DB;$$

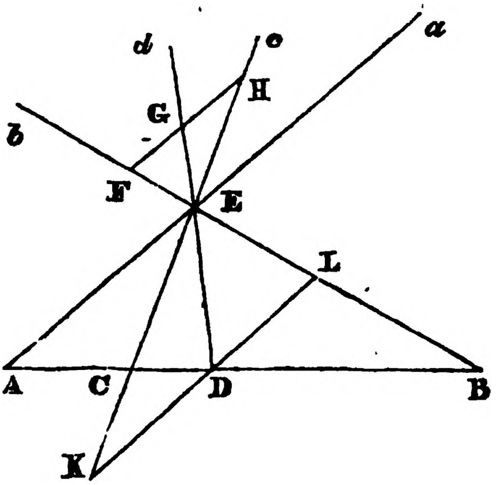
that is, $AB : BE :: AB-AE : AE-EB$.

Wherefore, AB, AE, and EB, are harmonic proportionals.

PROP. V.

5. **THEOREM.** *If from the extremities of a given straight line, which is divided harmonically, and from the points of division, straight lines be drawn, all passing through a given point, without the given line, any straight line, which is parallel to one of them, considered as the first, and is terminated by the second and fourth, shall be bisected by the third of them.*

Let $\overline{AB}$ be divided harmonically in C and D;



through any point E, without AB, let there be drawn the straight lines Aa, Cc, Dd, Bb, and $\overline{FH}$, drawn parallel to the first *Aa*, be terminated by *Cc* the second, and *Bb* the fourth: *FH* is bisected, in *G*, by *Dd*, the third.

Through *D*, draw *LDK* parallel to *Aa*, meeting *Bb* in *L*, and *Cc* in *K*; then (E. xv. 1. E. xxix. 1.) the △s *ACE*, *DCK*, are equiangular, as are, also, the △s *AEB*, *DLB*, the △s *FEG*, *DEL*, and the △s *GEH*, *KED*:

therefore (E. iv. 6.) $AE : AB :: DL : DB$;
and (*hyp.*) $AB : AC :: DB : DC$;
therefore (*ex æquo*) $AE : AC :: DL : DC$;
but (E. iv. 6.) $AE : AC :: DK : DC$;
therefore (E. ix. 5.) $DK = DL$;
and $ED : EG :: KD : GH :: DL : GF$;
$\therefore\ GH = GF$.

PROP. VI.

6. THEOREM. *If four given straight lines have a point common to them all, and if a straight line, drawn parallel to any one, considered as the first, and terminated by the second and fourth, be bisected by the third of them, any straight line, which cuts all the four given straight lines, shall be divided by them harmonically.*

Let the four given straight lines, *Aa*, *Cc*, *Dd*, *Bb*, (see the figure to Art. 5.) have the common point *E*, and let *FH*, drawn parallel to *Aa*, and terminated by *Cc* and *Bb*, be bisected by *Dd* in *G*; also, let $\overline{AB}$ cut the four given straight lines, in *A*, *C*, *D*, and *B*: *AB* is divided harmonically in *C* and *D*.

Through D, draw KL parallel to Aa, meeting Cc in K, and Bb in L; then

(E. iv. 6.) $EG : ED :: GH : KD :: GF : DL$;

and (*hyp.*) $GH = GF$; $\therefore$ (E. xiv. 5.) $KD = DL$.

And, since (E. xv. 1. E. xxix. 1.) the $\triangle$s AEB, DLB, are equiangular, as are, also, the $\triangle$s ACE, DCK,

(E. iv. 6.) $AB : AE :: DB : DL$,

and $AE : AC :: DK$ or $DL : DC$,

therefore (*ex æquo*) $AB : AC :: DB : DC$.

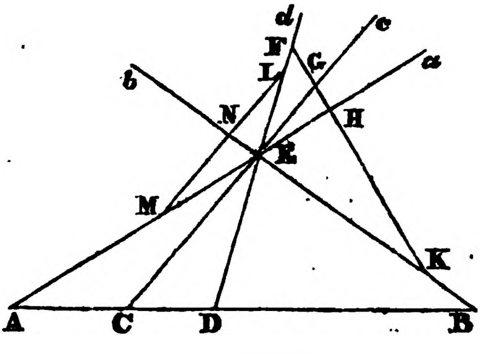
Wherefore, AB is cut harmonically in C and D.

7. Scholium. It is evident, that the above proposition furnishes a ready practical method, requiring no instrument but the parallel ruler, for finding an harmonic mean between two given unequal straight lines, and, also, a third harmonic proportional to them.

Prop. VII.

8. Theorem. *If four given straight lines, having a point common to them all, cut a fifth straight line harmonically, any other straight line, which is cut by the four given straight lines, shall be divided by them harmonically.*

Let the four straight lines Aa, Cc, Dd, Bb,



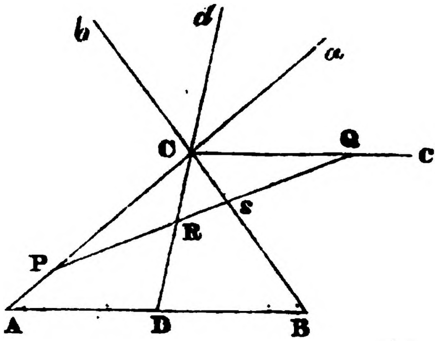
having the common point *E*, cut $\overline{AB}$ harmonically; and let *FK* be any other straight line cut by them, in the several points, *F*, *G*, *H*, *K*: *FK* is divided harmonically in *G* and *H*.

For, if *LM* be drawn parallel to *Cc*, it will (Supp. Part II. v. 5.) be bisected in *N*, by *Bb*; wherefore (Supp. Part II. vi. 5.) *FK* is cut harmonically in *G* and *H*.

PROP. VIII.

9. PROBLEM. *To draw four straight lines, so that any other straight line, which is cut by them all, shall be divided by them harmonically.*

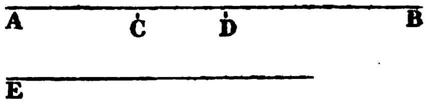
Take any three points *A*, *B* and *C*, which are not in the same straight line; join any two of them, as *A*, *B*; bisect *AB* in *D*, through *C* draw the straight lines *ACa*, *DCd*, *BCb*, and also, *Cc* parallel to *AB*: then (Supp. Part II. vi. 5.) shall any straight line *PQ*, which is cut by *Aa* in *P*, by *Dd* in *R*, by *Bb* in *S*, and by *Cc* in *Q*, be divided harmonically in *R* and *S*.



PROP. IX.

10. **PROBLEM.** *To find four geometrically proportional straight lines, the differences of which shall be harmonic proportionals.*

Assume any finite straight line AB; divide it (Supp. ii. 6. *Cor.* 1.) harmonically in C and D; find (E. xii. 6.) a fourth proportional E to BC, CA, and $AC+AD$: then are E, $E+AC$, $E+AC+AD$, and $E+AC+AD+AB$, the proportionals which were to be found.



For (*hyp.*) $BC : CA :: CA+AD : E$,
therefore (*componendo*)

$$AB : AC :: AC+AD+E : E,$$

and (*permutando*)

$$AB : AC+AD+E :: AC : E;$$

therefore (*componendo*)

$$AB+AC+AD+E : AC+AD+E :: AC+E : E;$$

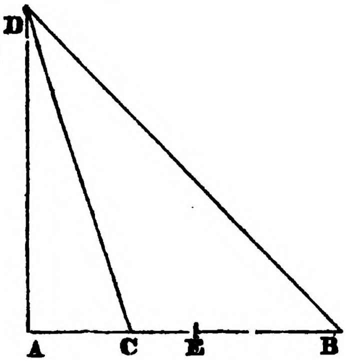
Wherefore, the four lines are proportionals, and (*constr.*) their differences AB, AD and AC, are harmonic proportionals.

11. COR. If the four proportionals, thus determined, be placed in the same straight line, it is manifest, that by dividing any given straight line similarly to it, the given line will be divided into four proportionals, the differences of which shall be harmonic proportionals.

Prop. X.

12. **Problem.** *To find a square, which shall be an harmonic mean proportional, between two given unequal squares.*

Let AB and AC be the sides of the two given squares: It is required to find a square, which shall be an harmonic mean proportional between $\overline{AB}^2$ and $\overline{AC}^2$.



From A, draw AD perpendicular to AB, and make AD equal to AB; join D, C, and D, B, and from AB cut off AE equal to a fourth proportional to DC, DB and AC: $\overline{AE}^2$ is the square, which was to be found.

For (*constr.*)

$$DC : DB :: AC : AE,$$

therefore (Supp. i. 6. *Cor.* 1.)

$$\overline{DC}^2 : \overline{DB}^2 :: AC^2 : AE^2;$$

(*constr.* and E. xlvii. 1.)

$$AB^2+AC^2 : 2AB^2 :: AC^2 : AE^2,$$

therefore (*convertendo*)

$$AB^2+AC^2 : AB^2-AC^2 :: AC^2 : AE^2-AC^2,$$

and (*permutando*)

$$AB^2 + AC^2 : AC^2 :: AB^2 - AC^2 : AE^2 - AC^2;$$

therefore (*dividendo*)

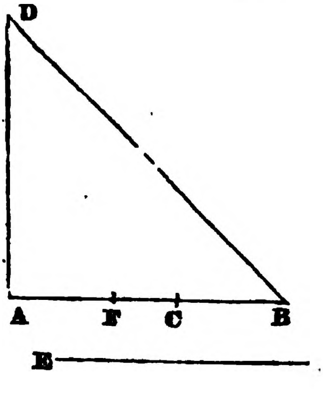
$$AB^2 : AC^2 :: AB^2 - AE^2 : AE^2 - AC^2.$$

Wherefore, AB^2, AE^2, and AC^2, are harmonic proportionals.

Prop. XI.

13. **Problem.** *To find a square, which shall be a third harmonic proportional to two given unequal squares.*

Let AB and AC be the sides of the two given squares: It is required to find a square, which shall be an harmonic third proportional to $\overline{AB}^2$ and $\overline{AC}^2$.



From A draw AD perpendicular to AB, and make AD equal to AB; join D, B; find (Supp. lxxv. 1.) the side E, of a square, which is equal to the excess of $\overline{DB}^2$ above $\overline{AC}^2$; and from AB cut off AF equal to a fourth proportional to E, AB and AC: $\overline{AF}^2$ is the square, which was to be found.

For (*constr.*) $E : AB :: AC : AF$, therefore (Supp. i. 6. *Cor.* 1.)

$$E^2 : AB^2 :: AC^2 : AF^2;$$

that is, (*constr.* and E. xlvii. 1.)

$$2AB^2 - AC^2 : AB^2 :: AC^2 : AF^2;$$

therefore (*dividendo*)

$$AB^2 - AC^2 : AB^2 :: AC^2 - AF^2 : AF^2;$$

and (*permutando*)

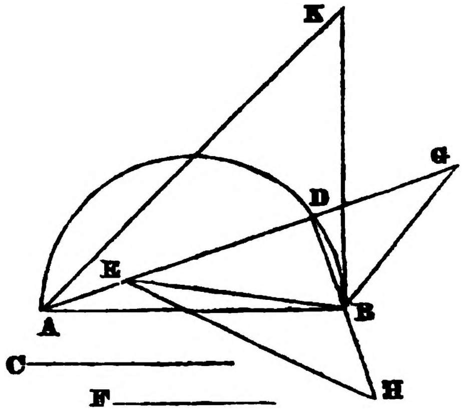
$$AB^2 - AC^2 : AC^2 - AF^2 :: AB^2 : AF^2.$$

Wherefore, AB^2, AC^2, and AF^2, are harmonic proportionals.

PROP. XII.

14. PROBLEM. *To find four geometrically proportional straight lines, the differences of the squares of which shall be harmonic proportionals.*

Assume any two unequal straight lines, *AB* and *C*; upon *AB*, the greater, as a diameter, describe the circle *ADB*, find (Supp. Part II. xi. 5.) a square which shall be a third harmonic proportional to $\overline{AB}^2$



and C^2, and place, in the circle ADB, BD equal to the side of that square; join A, D, and in DA, produced if necessary, take DE equal to C, and join B, E; find a fourth proportional, F, to AD, DB, and BE; produce AD to G, so that $DG=F$, and join B, G; produce, also, DB to H, so that $DH=BG$, and join E, H; lastly, draw BK perpendicular to AB, make it equal to EH, and join AK: Then are AK, EH, BG, and F, the four proportionals, which were to be found.

For (*constr.*)

$$AD : DB :: BE : F,$$

therefore (Supp. i. 6. *Cor.* 1.)

$$\overline{AD}^2 : \overline{DB}^2 :: \overline{BE}^2 : F^2,$$

and (*componendo*)

$$AD^2+DB^2 : DB^2 :: BE^2+F^2 : F^2;$$

therefore (*permutando*)

$$AD^2+DB^2 : BE^2+F^2 :: DB^2 : F^2,$$

and (*componendo*)

$$AD^2+DB^2+BE^2+F^2 : BE^2+F^2 :: DB^2+F^2 : F^2.$$

But (E. xlvii. 1.)

$$AD^2+DB^2=AB^2, \text{ and } BE^2+F^2=\overline{DE}^2+DB^2+F^2$$
$$=(\textit{constr.})\ DE^2+DB^2+\overline{DG}^2=DE^2+BG^2$$
$$=DE^2+\overline{DH}^2=\overline{EH}^2=\overline{BK}^2;$$
$$\therefore\ AD^2+DB^2+BE^2+F^2=AB^2+BK^2=\overline{AK}^2.$$

Wherefore, $AK^2 : EH^2 :: BG^2 : F^2$,

and (Supp. I. 6. *Cor.* 2.) $AK : EH :: BG : F$.

Therefore, AK, EH, BG and F, are geometrically proportional, and

$$AK^2-EH^2=AB^2,\ EH^2-BG^2=DE^2=C^2,$$
$$BG^2-F^2=BG^2-DG^2=BD^2;$$

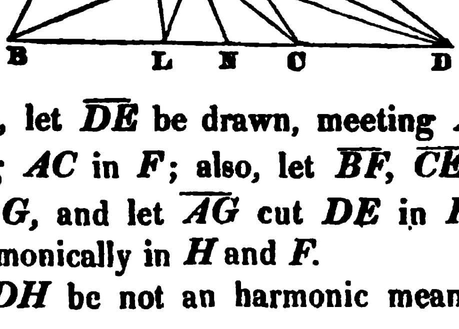
and (*constr.*) the squares of AB, C and BD, are harmonic proportionals.

15. **Cor.** If the four proportionals, thus determined, be placed in the same straight line, any given finite straight line, which is divided similarly to it, is divided into four geometric proportionals, the differences of the squares of which are harmonic proportionals.

Prop. XIII.

16. **Theorem.** *If the base of a triangle be produced, and if, from any point in the part produced, a straight line be drawn cutting the one side of the triangle and meeting the other, it shall be cut harmonically by the side of the triangle, and by the straight line drawn from the summit of the triangle, through the intersection of the two straight lines that join the angles at the base, and the points, in which the opposite sides are cut by the line itself so drawn.*

From any point D, in the base BC, produced, of



the $\triangle ABC$, let $\overline{DE}$ be drawn, meeting AB in E, and cutting AC in F; also, let $\overline{BF}$, $\overline{CE}$, cut one another in G, and let $\overline{AG}$ cut DE in H: DE is divided harmonically in H and F.

For, if DH be not an harmonic mean between

DE and *DF*, let *DK* be an harmonic mean between them; join *K*, *G*, and *G*, *D*, and *A*, *D*; produce *KG* to meet *BD* in *L*; and join *A*, *L*, and let *AL* cut *DE* in *M*. And, because (*hyp.*) *ED*, which cuts *GE*, *GK*, *GF*, *GD*, is divided harmonically in *K*, and *F*, *BD*, which cuts the same four lines, is also (Supp. Part II. vii. 5.) divided harmonically in *L*, and *C*; wherefore, likewise, *DE* is divided harmonically in *M* and *F*; so that *DM* is an harmonic mean between *DE* and *DF*; wherefore, *DM* is equal to *DK*, the greater to the less; which is impossible. Therefore, no line, but *DH*, is an harmonic mean between *DE* and *DF*.

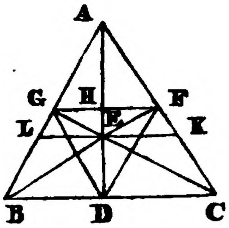
17. Cor. If *AG* be produced to meet the base *BC*, in *N*, *BD* is (Supp. Part II. vii. 5.) divided harmonically in *N* and *C*.

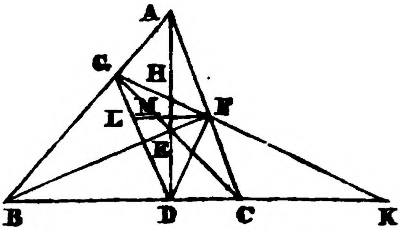
Prop. XIV.

18. Theorem. *If straight lines be drawn, from the extremities of the base of a triangle, through any the same point of the perpendicular, let fall from the summit on the base, the straight line joining the two points, in which they meet the sides, shall be divided, by the perpendicular, into two parts, that subtend equal angles, at the point where the perpendicular meets the base.*

Let *AD* be the perpendicular, drawn from the summit *A*, to the base *BC*, of the $\triangle ABC$; and through any point *E* of *AD*, let there be drawn $\overline{BEF}$, $\overline{CEG}$, meeting *AC*, and *AB*, in *F* and *G*, respectively; let *GF* be cut, by *AD* in *H*, and join *F*, *D*, and *G*, *D*: the $\angle FDH = \angle GDH$.

First, if GF be parallel to BC, through E draw





KEL also parallel to BC. Then

(E. ii. 6.) $BF : BE :: CG : CE$,
also (E. iv. 6.) $BF : BE :: FG : EL$,
and $CG : CE :: FG : EK$;
therefore (E. xi. 5. E. ix. 5.) $LE = EK$;

therefore (Supp. lxi. 1.) $GH = HF$; and HD is common to the two $\triangle$s GHD, FHD, and (*hyp.* and E. xxix. 1.) the $\angle$s GHD, FHD are right angles; wherefore, (E. iv. 1.) the $\angle FDH = \angle GDH$.

But if GF be not parallel to BC, let GF meet BC, produced, in K; and through F, draw FL parallel to BC, cutting AD in M, and meeting GD in L. Wherefore, (Supp. Part II. xiii. 5.) GK is cut harmonically in H and F, and (Supp. Part II. v. 5.) FL is bisected in M; it may be shewn, therefore, as in the former case, that the $\angle FDM = \angle LDM$; that is, the $\angle FDH = \angle GDH$.

A

SUPPLEMENT

TO THE

ELEMENTS OF EUCLID.

PART II.

BOOK VI.

Prop. I.

1. Theorem. *If two unequal arches of a circle be each of them less than a quadrant, the greater of them shall have to the less a greater ratio, than the supplement of the less has to the supplement of the greater.*

Let the arches *CD*, *CE*, of the circle *ABCE*, of which *CA* is a diameter, be each of them less than a quadrant:

$$\overset{\frown}{CE} : \overset{\frown}{CD} > \overset{\frown}{AD} : \overset{\frown}{AE}.$$

Since (*hyp.*) $\overset{\frown}{AE} > \overset{\frown}{CD}$, if $\overset{\frown}{DE}$ be added to each, (Supp. xi. 5.)

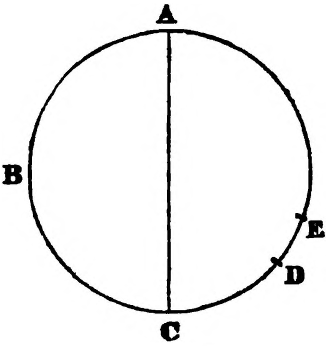
$$\overset{\frown}{AE} : \overset{\frown}{DC} > \overset{\frown}{AED} : \overset{\frown}{CDE},$$

therefore (Supp. vii. 5.)

$$\overset{\frown}{AE} : \overset{\frown}{AD} > \overset{\frown}{DC} : \overset{\frown}{CE},$$

and (Supp. vi. 5.)

$$\overset{\frown}{CE} : \overset{\frown}{CD} > \overset{\frown}{AD} : \overset{\frown}{AE}.$$



2. **Cor.** The greater of two angles has to the less a greater ratio, than the supplement of the less has to the supplement of the greater.

Prop. II.

3. **Theorem.** *If two unequal arches of a circle be each of them less than a quadrant, the tangent of the greater shall have to the tangent of the less, a greater ratio than the greater arch has to the less.*

Let the arches *AC* and *AD*, of the circle *ABC*, of which *AB* is a diameter, and *K* the centre, be each of them less than a quadrant; and let $\overline{KCE}$, $\overline{KDF}$, meet $\overline{AF}$, which touches the circle at *A*, in *E* and *F*, respectively,

$$\overline{AF} : \overline{AE} > \overset{\frown}{AD} : \overset{\frown}{AC}.$$

Through *C* draw *GCH* parallel to *AF*, meeting *KF* in *G*, and *KA* in *H*. Then (E. viii. 5.)

$$\triangle HKC : \triangle CKG < \text{sector } AKC : \triangle CKG;$$

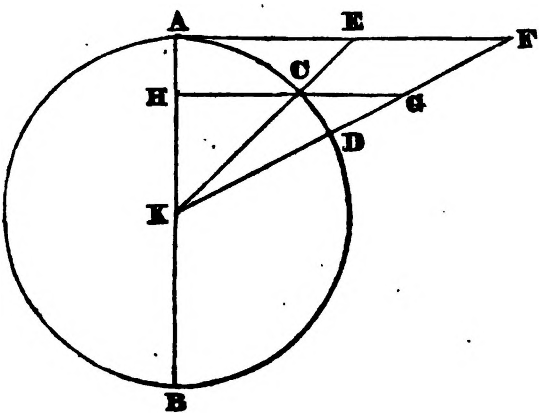
and sector $AKC : \triangle CKC$

$$< \text{sector } AKC : \text{sector } CKD;$$

$$\therefore \triangle HKC : \triangle CKG < \text{sector } AKC : \text{sector } CKD.$$

But (E. i. 6.)

$$\triangle HKC : \triangle CKG :: HC : CG,$$



and (E. xxxiii. 6.)

$$\text{sector } AKC : \text{sector } CKD :: \overset{\frown}{AC} : \overset{\frown}{CD}$$

$$\therefore HC : CG < \overset{\frown}{AC} : \overset{\frown}{CD},$$

$$\text{or } CG : HC > \overset{\frown}{CD} : \overset{\frown}{AC};$$

therefore, (Supp. x. 5.)

$$HG : HC > \overset{\frown}{AD} : \overset{\frown}{AC};$$

but (E. iv. 6.)

$$HG : HC :: AF : AE;$$

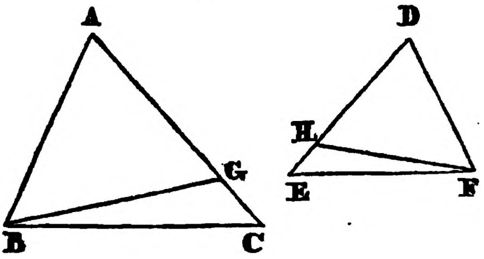
$$\therefore AF : AE > \overset{\frown}{AD} : \overset{\frown}{AC}.$$

Prop. III.

4. **Theorem.** *If two triangles have their vertical angles equal to one another, and have the two remaining angles in each, neither of which is obtuse,*

unequal, the side adjacent to the greater angle, in the one triangle, shall have to the base a less ratio, than that which the side adjacent to the corresponding angle, in the other triangle, has to the base.

In the two △s *ABC, DEF,* let the ∠ *A* be equal



to the ∠ *D*; and let the ∠ *ABC* be greater than the ∠ *DEF*; therefore (E. xxxii. 1.) the ∠ *ACB* is less than the ∠ *DFE*; and let none of the ∠s *ABC, ACB, DEF, DFE,* be obtuse;

$$AB : BC > DE : EF.$$

At the point *B* in *AB*, make the ∠ *ABG* equal to the ∠ *DEF*; wherefore, the two △s *ABG, DEF,* are (E. xxxii. 1.) equiangular; and since, (*hyp.*) the ∠ *DFE*, or *AGB*, is not obtuse; the ∠ *BGC* is not acute; wherefore (E. xxxii. 1.) the ∠ *BCG* is less than the ∠ *BGC*, and (E. xix. 1.) $BC > BG$; but (*constr.* and E. iv. 6.)

$$DE : EF :: AB : BG,$$

and, (E. viii. 5.)

$$AB : BG > AB : BC;$$

$$\therefore DE : EF > AB : BC,$$

$$\text{or } AB : BC < DE : EF.$$

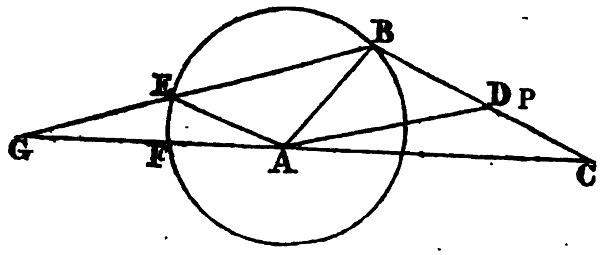
5. Cor. If the ∠ *DFH* be made equal to *ACB*, it may, in like manner be shewn, that

$$DF : FE < AC : CB.$$

PROP. IV.

6. **THEOREM.** *If, from the greater of two unequal sides of a given triangle, be cut off a part equal to the less, that segment shall have to the remaining segment, a ratio greater than the ratio which the angle adjacent to the remaining segment, has to the angle adjacent to the segment first cut off.*

Let the side AB, of the triangle ABC, be less



than the side BC, and from BC let there be cut off $BD = AB$:

Then $(BD : DC) > (\angle ACB : \angle ABC)$.

For draw $\overline{AD}$, and complete (E. xxxi. 1.) the ▱ $ADBE$; from the centre A, at the distance AB, describe the circle BEF, the circumference of which, since (E. xxxiv. 1. and *constr.*) $AE = BD$ or AB, will pass through E; produce CA to meet the circumference BEF in F, and BE produced in G; so that (E. xxix. 1.) the $\triangle$s GEA, GBC, are equiangular.

Then, since the sector AEF is less than the $\triangle$ AEG, and the sector AEB greater than the $\triangle$ AEB, therefore

$(\triangle AEG : \text{sector } AEF) > (\triangle AEB : \text{sector } AEB)$;

therefore (Supp. vii. 5.)

$(\triangle AEG : \triangle AEB) > (\text{sector } AEF : \text{sector } AEB)$:

But (E. i. 6.) $\triangle AEG : \triangle AEB :: GE : EB$, and (E. iv. 6. and E. xxxiv. 1.)

$$GE : EB :: BD : DC;$$

$$\therefore (BD : DC) > (\text{sector } AEF : \text{sector } AEB).$$

Also (E. xxxiii. 6.)

$$\text{sector } AEF : \text{sector } AEB :: \angle EAF : \angle EAB;$$

and (E. xxix. 1.)

$$\text{the } \angle EAF = \angle ACB, \text{ and the } \angle EAB = \angle ABC;$$

therefore

$$\text{sector } AEF : \text{sector } AEB :: \angle ACB : \angle ABC;$$

$$\therefore (BD : DC) > (\angle ACB : \angle ABC).$$

7. COR. If any part BP be taken of BC, that is greater than AB, then, much more, is

$$(BP : PC) > \angle (ACB : \angle ABC).$$

PROP. V.

8. THEOREM. *If two unequal arches of a circle have a common extremity, the one being a part of the other, and if, from their two other extremities, any two parallels be drawn, to meet the diameter passing through the common extremity, the greater arch shall have to the less, a greater ratio than the parallel terminating the greater arch, has to the parallel terminating the less.*

Let BC, BD, be any two unequal arches of the circle $ABCD$, of which K is the centre, and BKA a diameter; and let CF, DG be any two parallels meeting BA in F and G;

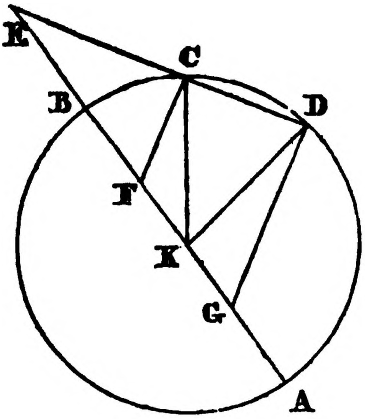
$$\widehat{BD} : \widehat{BC} > DG : CF.$$

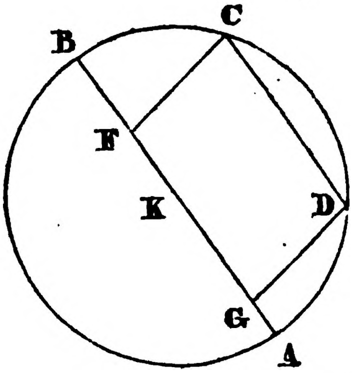
Join D, C; and, first, if DC be parallel to AB; (*hyp*. and E. xxxiv. 1.)

$$\overline{FC} = \overline{GD}, \text{ and } \widehat{BD} > \widehat{BC};$$

$$\therefore \widehat{BD} : \widehat{BC} > \overline{GD} : \overline{FC}.$$

But, if DC be not parallel to AB, DC and AB,





produced, will meet either towards A, or towards B: if they meet towards A, $FC > GD$,

$$\text{and } \therefore \overset{\frown}{BD} : \overset{\frown}{DC} > GD : FC:$$

but if they meet towards B, let them meet in the point E, and join K, C, and K, D; therefore (E. viii. 5.)

$$\text{sector } BCK : \text{sector } CKD < \triangle ECK : \text{sector } CKD,$$
$$\text{and } \triangle ECK : \text{sector } CKD < \triangle ECK : \triangle CDK;$$

therefore

$$\text{sector } BCK : \text{sector } CKD < \triangle ECK : \triangle CDK,$$

that is, (E. xxxiii. 6.)

$$\overset{\frown}{BC} : \overset{\frown}{CD} < \triangle ECK : \triangle CDK \text{ or } \overline{EC} : \overline{CD},$$

therefore (Supp. vi. 5.)

$$\overset{\frown}{CD} : \overset{\frown}{BC} > CD : EC,$$

therefore (Supp. x. 5.)

$$\overset{\frown}{BD} : \overset{\frown}{BC} > DE : EC,$$

but (E. iv. 6.)

$$ED : EC :: DG : CF;$$
$$\therefore \overset{\frown}{BD} : \overset{\frown}{BC} > DG : CF.$$

9. COR. 1. If CF and DG be each perpendicular

to AB, they are the sines of $\overset{\frown}{BC}$ and $\overset{\frown}{BD}$: wherefore, the greater of two arches of a circle has to the less, a greater ratio, than the sine of the greater has to the sine of the less.

10. Cor. 2. The greater of two arches of a circle has to the less a greater ratio, than the chord of the greater has to the chord of the less.

11. Cor. 3. If from any point, in the diameter produced, of a given circle, a straight line be drawn, cutting the circle, the part of it without, has to the part of it within the circle, a greater ratio than the arch between the diameter and the cutting line, has to the arch cut off.

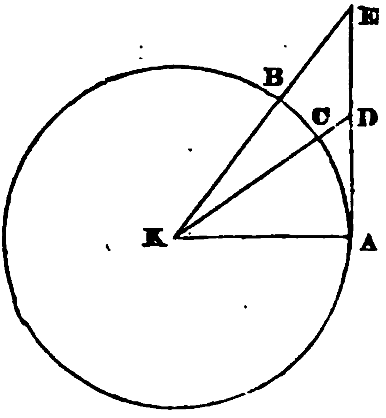
For it was shewn, that

$$EC : CD > \overset{\frown}{BC} : \overset{\frown}{CD}.$$

Prop. VI.

12. Theorem. *The tangent of the greater of two arches of a circle has to the tangent of the less, a greater ratio than the secant of the greater, has to the secant of the less.*

Let AB and AC, of which AB is the greater, be



arches of the circle ABC, the centre of which is K;

and let KC, KB, produced, meet in D and E respectively, AE, which touches the circle at A:

$$AE : AD > KE : KD.$$

For in the right-angled △s KAE, KAD, the

$$\angle KDA > \angle KEA;$$

therefore (Supp. Part II. iii. 6.)

$$AD : DK < AE : EK,$$

$$\text{or } AE : EK > AD : DK,$$

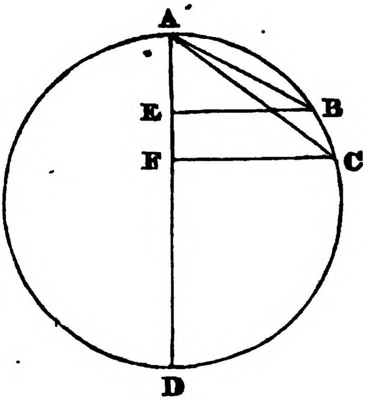
therefore (Supp. vii. 6.)

$$AE : AD > EK : DK.$$

PROP. VII.

13. **THEOREM.** *The sine of the greater of two arches of a circle has to the sine of the less, a less ratio than the chord of the greater has to the chord of the less.*

Let $\overparen{AB}$, $\overparen{AC}$, be two unequal arches of the circle



ADB, of which AD is the diameter, let A, B, and A, D, be joined, and let BE and CF be drawn each of them perpendicular to AD;

$$CF : BE < \overline{AC} : \overline{AB}.$$

For, in the right-angled $\triangle$s AFC, AEB, the $\angle\, BAE$ is greater than the $\angle\, CAF$; therefore (Supp. Part II. iii. 6.)

$$FC : CA < EB : BA,$$

(E. vii. 5.) $FC : EB < \overline{CA} : \overline{BA}.$

Prop. VIII.

14. **Theorem.** *The chord of the greater of two arches of a circle, has to the chord of the less, a less ratio than the versed sine of the greater, has to the versed sine of the less.*

The same hypothesis, and construction, being made as in Art. 13.

$$\overline{AC} : \overline{AB} < AF : AE.$$

For (Supp. Part II. iii. 6.)

$$EA : AB < FA : AC,$$

or $FA : AC > EA : AB;$

therefore (Supp. vii. 5.)

$$FA : EA > AC : AB;$$

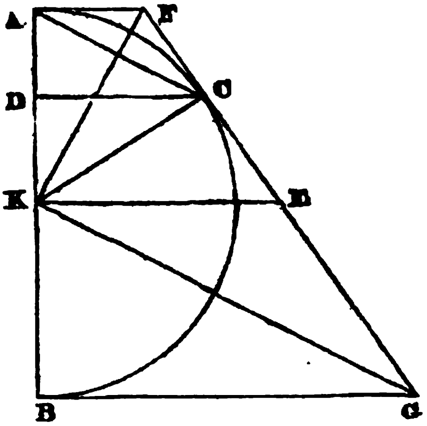
that is, $AC : AB < FA : AE.$

15. **Cor.** The sine of the greater of two arches of a circle has to the sine of the less, a less ratio than the versed sine of the greater has to the versed sine of the less.

Prop. IX.

16. **Theorem.** *If the semi-circumference of a given circle be divided into any two unequal arches, the semi-circumference has to the greater arch, a greater ratio than the less arch has to its sine.*

Let $\widehat{AEB}$, the semi-circumference of a circle, of which *K* is the centre, and *AD* a diameter, be



divided into two unequal parts in *C*, and let $\widehat{BC} > \widehat{AC}$; also, let *CD* be drawn perpendicular to *AB*;

$$\widehat{ACB} : \widehat{CB} > \widehat{AC} : CD.$$

For join *K*, *C*; draw *AF*, *FG*, *BG*, touching the circle in *A*, *C*, and *B*, respectively; also, draw *KE* parallel to *DC*. Wherefore, (E. xxxvi. 3.)

$$FA = FC,\ GB = GC,$$

and (Supp. Part II. xxii. 1.)

$$KE = EG = EF;$$

wherefore, the ∠ *FKG* is an angle in a semi-circle; that is, it is a right angle; and (E. xviii. 3.) *KC* is at right angles to *FG*; therefore (E. viii. 6.) the △s *KCF*, *KCG*, are equiangular, as are, also, (*constr.* and E. xxix. 1.) the △s *KDC*, *KCE*; therefore (E. iv. 6.)

$$KE : KC :: KC : CD,$$

$$\text{and } KC : CG :: CF : KC;$$

therefore (*ex æquo*)

$$KE : CG :: CF : CD,$$

and (E. iv. 5.)

$$4\,KE : 2\,CG :: 2\,CF : CD;$$

that is, $2\,CG + 2\,CF : 2\,CG :: 2\,CF : CD$.

$$\text{But } 2\,CG = GC + GB > \overset{\frown}{BC},$$

$$\text{and } 2\,CF = FC + FA > \overset{\frown}{AC};$$

wherefore, (Supp. xi. 5. *Cor.*)

$$\overset{\frown}{ACB} : \overset{\frown}{CB} > \overset{\frown}{AC} : \overline{CD}.$$

17. Cor. Join A, C, and B, C; then (Supp. Part II. v. 6. *Cor.* 2.)

$$\overset{\frown}{AC} : \overset{\frown}{CB} > \overline{AC} : \overline{BC};$$

therefore (Supp. x. 5.)

$$\overset{\frown}{ACB} : \overset{\frown}{CB} > BC + AC : BC.$$

If, therefore, X be a fourth proportional to $\overset{\frown}{ACB}$, $\overset{\frown}{CB}$ and $\overset{\frown}{AC}$, X is greater than a fourth proportional to $(AC+BC)$, BC and AC, and it is less than a fourth proportional to AB, BC and AC; for

$$AB : BC :: AC : CD.$$

PROP. X.

18. THEOREM. *If from any point in the diameter, produced, of a semi-circle, two straight lines be drawn to the concave circumference, the greater of them shall have to the less, a less ratio than that, which the greater of the two arches cut off by them, has to the less.*

From the point A, in the diameter CB produced, of the semi-circle $BEDC$, let $\overline{AD}$ and $\overline{AE}$ be drawn, meeting the concave circumference in D and E, and cutting off the arches $\overset{\frown}{FD}$ and $\overset{\frown}{GE}$, of which FD is the greater:

$$AD : AE < \widehat{FD} : \widehat{GE}.$$

For (E. xv. 3.) $\overline{FD} > \overline{GE}$,
and (E. viii. 3.) $AG > AF$;
therefore (Supp. xii. 5.)

$$FD : GE > AD : GE + AF,$$

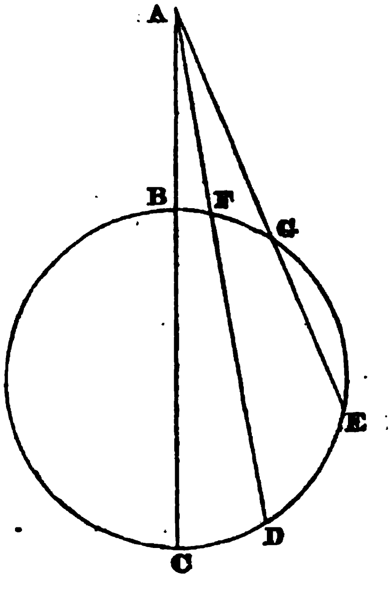
and, therefore,

$$FD : GE > AD : AE;$$

but (Supp. Part II. v. 6. Cor. 2.)

$$\widehat{FD} : \widehat{GE} > FD : GE;$$

$$\therefore \widehat{FD} : \widehat{GE} > AD : AE.$$



19. **Cor.** Since (E. xxxvi. 3. Cor. and E. xvi. 6.)

$$AD : AE :: AG : AF;$$

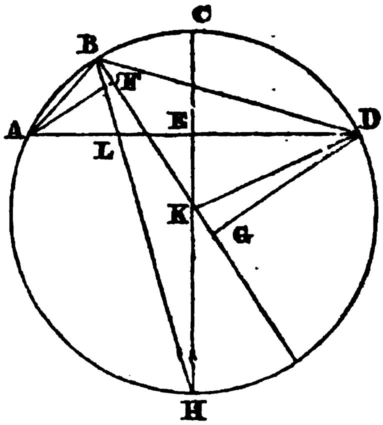
therefore, $\widehat{AG} : \widehat{AF} < \widehat{FD} : \widehat{GE}$.

PROP. XI.

20. **Theorem.** *If any given arch of a circle,*

and its chord, be each of them divided into two unequal parts, by the same straight line drawn from the centre of the circle, the greater segment of the arch shall have to the less a greater ratio, than that which the greater segment of the chord has to the less.

Let K be the centre of the circle $ACDH$, and let



KB divide $\overset{\frown}{ABD}$ into two unequal parts, in the point B, of which $\overset{\frown}{AB}$ is the less, and let it cut $\overline{AD}$ in E; if the diameter HKC be drawn at right angles to AD, it will bisect (E. iii. 3. E. xxx. 3.) both AD and $\overset{\frown}{ABD}$; and since $\overset{\frown}{AB} < \overset{\frown}{BD}$, KC will fall between KB and KD, and therefore $DE > EA$; then $\overset{\frown}{DB} : \overset{\frown}{BA} > \overline{DE} : \overline{EA}$.

From A and D, draw AF, and DG perpendicular to BK; therefore (Supp. Part II. v. 6. Cor. 1.)

$$\overset{\frown}{DB} : \overset{\frown}{BA} > DG : AF;$$

and (E. iv. 6.) $DG : AF :: DE : EA$;

$$\therefore \overset{\frown}{DB} : \overset{\frown}{BA} > DE : EA.$$

21. **Cor.** If, from the extremity H, of the diameter CKH, $\overline{HB}$ be drawn, cutting AD in L,

$$\overset{\frown}{DB} : \overset{\frown}{BA} > \overline{DL} : \overline{LA}.$$

For draw AB, BD, and since $\overset{\frown}{AH} = \overset{\frown}{DH}$, the $\angle ABL = \angle DBL$ (E. xxvii. 3.);

$$\therefore \text{(E. iii. 6.)}\ DL : LA :: DB : \overline{BA},$$

and (Supp. Part II. v. 6. *Cor.* 2.)

$$\overset{\frown}{DB} : \overset{\frown}{BA} > DB : BA;$$

$$\therefore \overset{\frown}{DB} : \overset{\frown}{BA} > DL : LA.$$

Prop. XII.

22. **Problem.** *If the two angles, at the base of a triangle be unequal, the greater has to the less, a greater ratio than that, which the side opposite to the greater has to the side opposite to the less.*

For, if a circle be described about the triangle, the angles at the base will (E. xxxiii. 6.) be to each other, as the arches which subtend them, and of which the two sides are the chords; and (Supp. Part II. v. 6. *Cor.* 2.) the greater arch has to the less, a greater ratio than the chord of the greater has to the chord of the less: Wherefore, also, the greater angle has a greater ratio to the less, than the side opposite to the greater has to the side opposite to the less.

Prop. XIII.

23. **Theorem.** *If the two angles at the base of a triangle be unequal, and if a straight line be*

drawn bisecting the vertical angle, and cutting the base, of the two angles, at the base, the greater has to the less, a greater ratio than that which the greater segment of the base has to the less.

For, if a circle be described about the triangle, the angles at the base will (E. xxxiii. 6.) be to each other as the arches, which subtend them; and (Supp. Part II. xi. 6. *Cor.*) the greater arch has to the less a greater ratio than the greater segment of the base has to the less: Wherefore, also, the greater of the two angles at the base has to the less, a greater ratio, than that which the greater segment of the base has to the less.

PROP. XIV.

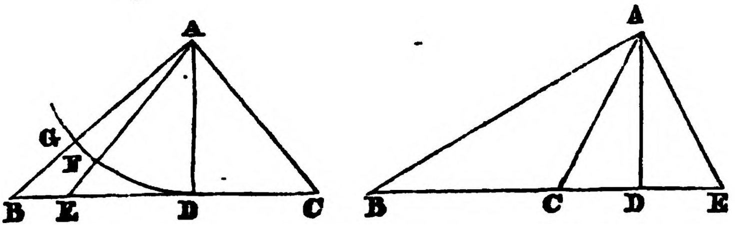
24. THEOREM. *If a straight line be drawn from the summit of a given isosceles triangle, which divides the angle, at the summit, into two unequal parts, it shall also divide the base into two unequal parts; and the greater segment of the angle, at the summit, shall have to the less a greater ratio, than that which the greater segment of the base has to the less.*

For if the straight line, drawn from the summit, be produced to meet the circumference of a circle, described about the triangle, it will divide the arch, subtended by the base into two unequal arches, of which the greater has (Supp. Part II. xi. 6.) to the less a greater ratio, than the greater of the two segments of the base has to the less: and (E. xxxiii. 6.) these unequal arches are to each other as the segments of the angle, at the summit: Wherefore, the greater segment, of the angle at the summit, has to the less, a greater ratio than that which the greater segment of the base has to the less.

PROP. XV.

25. THEOREM. *If from the summit of a given triangle, a perpendicular be drawn to the base, produced if it be necessary, of the angles which it makes with the other two sides, the greater shall have to the less a less ratio, than that, which the greater segment of the base, or the base produced, has to the less.*

Let ABC be the given triangle, and AD the perpendicular, drawn from the summit A, to the base BC, produced if it be necessary:



$$BD : DC > \angle BAD : \angle CAD.$$

In CD produced, take DE equal to DC; join A, E; and, first, if AD fall within the $\triangle ABC$, from the centre A, at the distance AD, describe the circle DFG; then (Supp. Part II. ii. 6.)

$$BD : ED \text{ or } DC > \widehat{GD} : \widehat{FD};$$

and (E. xxxiii. 6.)

$$\widehat{GD} : \widehat{FD} :: \angle BAD : \angle EAD \text{ or } \angle CAD;$$

$$\therefore BD : DC > \angle BAD : \angle CAD.$$

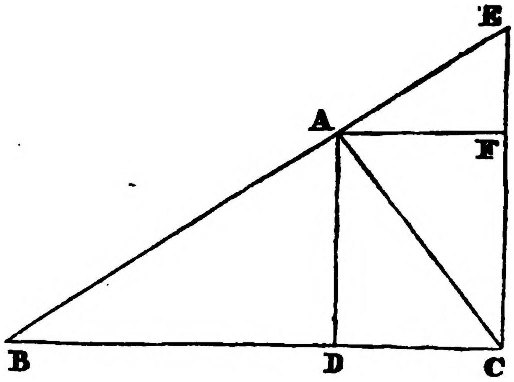
But, if *AD* fall without the $\triangle ABC$, then (*constr.* and E. iv. 1.) $\angle EAD = \angle CAD$, and, by the former case,

$$BD : DC > \angle BAD : \angle CAD.$$

PROP. XVI.

26. THEOREM. *If a perpendicular to the base of a scalene triangle, drawn from its summit, fall within the triangle, of the angles at the base, the greater has to the less, a less ratio, than that which the greater segment of the base has to the less.*

Let *AD*, drawn from the summit *A*, of the $\triangle ABC$, to the base *BC*, fall within the triangle, and let *AB* be greater than *AC*, and therefore (E. xviii. 1.) $\angle ACB > \angle ABC$; and it is manifest from E. xlvii. 1., that $BD > DC$: then



$$DB : DC > \angle ACB : \angle ABC.$$

From *C* draw *CE* parallel to *DA*, and meeting *BA* produced in *E*; also, from *A*, draw *AF* parallel to *BC*, and let it meet *CE* in *F*. Then, *AF* is perpendicular to CE, and $\angle CAF > \angle EAF$, because (E. xxix. 1.) $\angle CAF = \angle ACB$, and $\angle EAF = \angle ABC$, and (*hyp.*) $\angle ACB > \angle ABC$;

therefore (Supp. Part II. xv. 6.)

$$CF : FE > \angle CAF : \angle EAF;$$

and (E. ii. 6.)

$$CF : FE :: BA : AE :: BD : DC;$$

$$\therefore DB : DC > \angle CAF : \angle EAF,$$

that is, $DB : DC > \angle ACB : \angle ABC$.

THE END.

A

GENERAL INDEX,

EXHIBITING THE ENUNCIATIONS OF ALL THE PROPOSITIONS.

PART I. BOOK I.

I. PROB. A GIVEN plane rectilineal angle being divided into any number of equal angles, to divide the half of it into the same number of angles, all equal to one another.

II. PR. From the vertex of a given scalene triangle, to draw, to the base, a straight line which shall exceed the less of the two sides, as much as it is itself exceeded by the greater.

III. PR. In a straight line given in position, but indefinite in length, to find a point, which shall be equidistant from each of two given points, either on contrary sides, or both on the same side of the given line, and in the same plane with it; but not situated in a perpendicular to it.

IV. TH. If the three sides of a given triangle be bisected, the perpendiculars drawn to the sides, from the three several bisections, shall all meet in the same point: And that point is equidistant from the three angular points of the given triangle

V. PR. To find a point, in a given plane, which shall be equidistant from three given points in the plane, that are not all in the same straight line.

VI. TH. There cannot be drawn more than two equal straight lines, to another straight line, from a given point without it.

VII. Th. The perpendicular let fall from the obtuse angle of an obtuse-angled triangle, or from any angle of an acute-angled triangle, upon the opposite side, falls within that side: But the perpendicular drawn to either of the sides containing the obtuse angle of an obtuse-angled triangle, from the angle opposite, falls without that side.

VIII. Th. If a straight line, meeting two other straight lines, makes the two interior angles on the same side of it not less than two right angles, these lines shall never meet on that side, if produced ever so far.

IX. Th. The three sides of a triangle taken together, exceed the double of any one side, and are less than the double of any two sides.

X. Pr. Any side of a triangle is greater than the difference between the other two sides.

XI. Th. Any one side of a rectilineal figure is less than the aggregate of the remaining sides.

XII. Th. The two sides of a triangle are together, greater than the double of the straight line which joins the vertex and the bisection of the base.

XIII. Th. The two sides of a triangle are, together, greater than the double of the straight line drawn from the vertex to the base, bisecting the vertical angle.

XIV. Th. If a trapezium and a triangle stand upon the same base, and on the same side of it, and the one figure fall within the other, that which has the greater surface shall have the greater perimeter.

XV. Pr. One of the angles at the base of a triangle, the base itself, and the aggregate of the two remaining sides, being given, to construct the triangle.

XVI. Th. If two right-angled triangles have the three angles of the one equal to the three angles of the other, each to each, and if a side of the one be equal to the perpendicular let fall from the right angle upon the hypotenuse of the other, then shall a side of this latter triangle be equal to the hypotenuse of the former.

XVII. Th. If the sides of any given equilateral and equiangular figure of more than four sides, be produced so as to meet, the straight lines, joining their several intersections, shall contain an equilateral and equiangular figure, of the same number of sides as the given figure.

XVIII. Th. If two opposite sides of a quadrilateral figure be equal to one another, and the two remiaining sides be also equal to one another, the figure is a parallelogram.

XIX. Th. Every parallelogram which has one angle a right angle, has all its angles right angles.

XX. Pr. To trisect a right angle; i. e. to divide it into three equal parts.

XXI. Pr. Hence, to trisect a given rectilineal angle, which is the half, or the quarter, or the eighth part, and so on, of a right angle.

XXII. Pr. In the hypotenuse of a right-angled triangle, to find a point, the perpendicular distance of which from one of the sides, shall be equal to the segment of the hypotenuse between the point and the other side.

XXIII. Pr. In the base of a given acute-angled triangle, to find a point, through which if a straight line be drawn perpendicular to one of the sides, the segment of the base, between that side and the point, shall be equal to the segment of the perpendicular, between the point and the other side produced.

XXIV. Pr. From a given isosceles triangle to cut off a trapezium, which shall have the same base as the triangle, and shall have its three remaining sides equal to each other.

XXV. Pr. To draw to a given straight line, from a given point without it, another straight line which shall make with it an angle equial to a given rectilineal angle.

XXVI. Th. If all the angles but one of any rectilineal figure, be together, equal to all the angles but one, of another rectilineal figure having the same number of sides, the remaining angle of the one figure, shall be equal to the remaining angle of the other: And, conversely, if an angle in the one figure be equal to an angle in the other, the remaining angles of the one shall be equal, together, to the remaining angles of the other.

XXVII. Th. The angle at the base of an isosceles triangle is equal to, or is less, or greater, than the half of the vertical angle, accordingly as the triangle is a right-angled, an obtuse-angled, or an acute-angled triangle.

XXVIII. Th. If either of the equal sides of an isosceles triangle be produced, towards the vertex, the straight line, which bisects the exterior angle, shall be parallel to the base.

XXIX. Th. The distance of the vertex of a triangle from the bisection of its base, is equal to, greater than, or less than the half of the base, accordingly as the vertical angle is a right, an acute, or an obtuse angle.

XXX. Pr. Upon a given finite straight line, as a *diameter*, to describe a square.

XXXI. Th. If either of the acute angles of a given right-angled triangle be divided into any number of equal angles, then, of the segments of the base, subtending those equal angles, the nearest to the right angle is the least; and, of the rest, that which is nearer to the right angle is less than that which is more remote.

XXXII. Th. If either angle at the base of a triangle be a right angle, and if the base be divided into any number of equal parts, that which is adjacent to the right angle shall subtend the greatest angle at the vertex; and, of the rest, that which is nearer to the right angle shall subtend, at the vertex, a greater angle than that which is more remote.

XXXIII. Pr. To trisect a given finite straight line.

XXXIV. Pr. To describe a triangle which shall have its three sides, taken together, equal to a given finite straight line, and its three angles equal to three given angles, each to each; the three given angles being together equal to two right angles.

XXXV. Th. If, in the sides of a given square, at equal distances from the four angular points, four other points be taken, one in each side, the figure contained by the straight lines which join them, shall also be a square.

XXXVI. Th. If the opposite angles, of a quadrilateral figure be equal to each other, the figure shall be a parallelogram.

XXXVII. Pr. In a given square to inscribe an equilateral triangle, having one of its angular points upon one of the angular points of the square, and its two remaining angular points one in each of two adjacent sides of the square.

XXXVIII. Th. If, at the extremities of the base of a given triangle, two straight lines be drawn, both above the base, and each of them equal to the adjacent side, and making with it an angle equal to the vertical angle of the triangle; then, if two straight lines, let fall from the extremities of the two so drawn, make, with the base produced, two angles that are equal each of them to the vertical angle, they shall cut off equal segments from the base produced.

XXXIX. Th. If four straight lines cut each other, without including space, but so as to make three internal angles, towards the same parts, which together are less than four right angles, the two lines, which are not joined, shall meet, if produced far enough.

XL. Pr. To inscribe a square in a given right-angled isosceles triangle.

XLI. Pr. To find a point, in either of the equal sides of a given isosceles triangle, from which, if a straight line be drawn, perpendicular to that side, so as to meet the other side produced, it shall be equal to the base of the triangle.

XLII. Th. The diameters of a parallelogram bisect each other.

XLIII. Th. If in two opposite sides of a parallelogram two points be assumed, one in each of those sides equidistant from two opposite angles of the figure, and if two other points be likewise assumed, in the two other opposite sides, equidistant from the same two angles, the figure, contained by the straight lines joining the four points so assumed, shall be a parallelogram.

XLIV. Th. If any number of parallelograms be inscribed in a given parallelogram, the diameters of all the figures shall cut one another in the same point.

XLV. Th. The diameters of an equilateral four-sided plane rectilineal figure bisect one another at right angles.

XLVI. Th. The diameters of a rectangle are equal to one another.

XLVII. Pr. To inscribe a square in a given equilateral four-sided figure.

XLVIII. Th. If two opposite sides of a parallelogram be divided each into the same number of equal parts, the straight lines, joining the opposite points of division, shall also divide the diameter of the parallelogram into the same number of equal parts.

XLIX. Pr. To divide a given finite straight line into any given number of equal parts.

L. Pr. Upon a given finite straight line to describe an equilateral and equiangular octagon.

LI. Th. If either diameter of a parallelogram be equal to a side of the figure, the other diameter shall be greater than any side of the figure.

LII. Pr. From a given point to draw a straight line cutting two parallel straight lines, so that the part of it, intercepted between them, shall be equal to a given finite straight line, not less than the perpendicular distance of the two parallels.

LIII. Th. If, from the summit of the right angle of a scalene right-angled triangle, two straight lines be drawn, one perpendicular to the hypotenuse, and the other bisecting it, they shall contain an angle equal to the difference of the two acute angles of the triangle.

LIV. Pr. To bisect a parallelogram by a straight line drawn through a given point in one of its sides.

LV. Th. A trapezium, which has two of its sides parallel, is the half of a rectangle between the same parallels, and having its base equal to the aggregate of the two parallel sides of the trapezium.

LVI. Pr. Any two parallelograms having been described on two sides of a given triangle, to apply, to the remaining side, a parallelogram, which shall be equal to their aggregate.

LVII. Pr. A plane rectilineal figure of any number of sides being given, to find an equal rectilineal figure, which shall have the number of its sides less, or greater, by one, than that of the given figure.

LVIII. Th. The diameters of any parallelogram divide it into four equal triangles.

LIX. Pr. If two triangles have the two adjacent sides of a parallelogram for their bases, and have their common vertex situated in the diameter, or in the diameter produced, they shall be equal to one another.

LX. Th. Of all triangles, which are between the same parallels, that which stands on the greatest base is the greatest.

LXI. Th. The straight line, joining the vertex and the bisection of the base of any triangle, bisects every other straight line that is parallel to the base, and is terminated by the two remaining sides of the triangle, or those sides produced.

LXII. Th. If two opposite sides of a trapezium be parallel to one another, the straight line, joining their bisections, bisects the trapezium.

LXIII. Pr. To bisect a given trapezium by a straight line drawn from any of its angles.

LXIV. Pr. To bisect a given triangle by a straight line drawn through a given point in any one of its sides.

LXV. Pr. Equal triangles, which have their bases in the same straight line and which are between the same parallels, stand upon equal bases.

LXVI. Pr. To describe a parallelogram, the surface and perimeter of which shall be respectively equal to the surface and perimeter of a given triangle.

LXVII. Th. The two triangles formed by drawing straight lines, from any point within a parallelogram, to the extremities of either pair of opposite sides, are, together, half of the parallelogram.

LXVIII. Th. If two sides of a trapezium be parallel, the triangle contained by either of the other sides, and the two straight lines drawn from its extremities to the bisection of the opposite side, is the half of the trapezium.

LXIX. Th. The triangle contained by the straight lines joining the points of the bisection of the three sides of a given triangle, is one-fourth part of the given triangle, and is equiangular with it.

LXX. Pr. To describe a parallelogram, which shall be of a given altitude, and equiangular with, and also equal to, a given parallelogram.

LXXI. Th. If there be any number of rectilineal figures, of which the first is greater than the second, the second than the third, and so on, the first of them shall be equal to the last together with the aggregate of all the differences of the figures.

LXXII. Pr. To find a rectangle, which shall have one of its sides equal to a given finite straight line, and which shall be equal to the excess of the greater of two given rectilineal figures above the less.

LXXIII. Th. If two right-angled triangles have two sides of the one equal to two sides of the other, each to each, the triangles shall be equal, and similar to each other.

LXXIV. Pr. To find a square which shall be equal to any number of given squares.

LXXV. Pr. Two unequal squares being given, to find a third square, which shall be equal to the excess of the greater of them above the less.

LXXVI. Th. If the side of a square be equal to the diameter of another square, the former square shall be the double of the latter.

LXXVII. Th. In any right-angled triangle, the square which is described on the side subtending the right angle, as a diameter, is equal to the squares described upon the other two sides, as diameters.

Book II.

Prop. I. Th. If two given straight lines be divided, each into any number of parts, the rectangle contained by the two straight lines, is equal to the rectangles contained by the several parts of the one and the several parts of the other.

II. Th. If a straight line be divided into two unequal parts, in two different points, the rectangle contained by the two parts, which are the greatest and the least, is less than the rectangle contained by the other two parts; the squares of the two former parts, together, are greater than the squares of the two latter, taken together; and the difference between the squares of the former and the squares of the latter, is the double of the difference between the two rectangles.

III. Th. In any isosceles triangle, if a straight line be drawn from the vertex to any point in the base, the square upon this line, together with the rectangle contained by the segments of the base, is equal to the square upon either of the equal sides.

IV. Th. The rectangle contained by the aggregate and the difference of two unequal straight lines is equal to the difference of their squares.

V. Th. The square of the excess of the greater of two given straight lines above the less, is less than the squares of the two lines, by twice the rectangle contained by them.

VI. Th. The squares of any two unequal straight lines are, together, greater than twice the rectangle contained by those lines.

VII. Th. If a straight line be divided into five equal parts, the square of the whole line is equal to the square of the straight line, which is made up of four of those parts, together with the square of the straight line which is made up of three of those parts.

VIII. Pr. Upon a given straight line, as an hypotenuse, to describe a right-angled triangle, such that the hypotenuse, together with the less of the two remaining sides, shall be the double of the greater of those sides.

IX. Th. In any triangle, the squares of two sides are, together, the double of the squares of half the base, and of the straight line joining its bisection and the opposite angle.

X. Th. The squares of the sides of any parallelogram are, together, equal to the squares of its diameters taken together.

XI. Th. If either diameter of a parallelogram be equal to one of the sides about the opposite angle of the figure, its square shall be less than the square of the other diameter, by twice the square of the other side about that opposite angle.

XII. Th. If two sides of a trapezium be parallel to each other, the squares of its diagonals are, together, equal to the aggregate of the squares of its two sides, which are not parallel, and of twice the rectangle of its parallel sides.

XIII. Th. The square of the base of an isosceles triangle is the double of the rectangle contained by either side, and by the straight line intercepted between the perpendicular, let fall upon it from the opposite angle, and the extremity of the base.

XIV. Th. If from any point, in the circumference of the greater of two given concentric circles, two straight lines be drawn to the extremities of any diameter of the less, their squares shall be, together, the double of the squares of the two semi-diameters of the two given circles.

Book III.

Prop. I. Th. If two circles cut each other, the straight line joining their two points of intersection is bisected, at right angles, by the straight line joining their centres.

II. Pr. Through a given point within a circle, which is not the centre, to draw a chord which shall be bisected in that point.

III. Th. If two isosceles triangles be of equal altitudes, and the side of one be equal to the side of the other, their bases shall be equal.

IV. Th. Any two chords of a circle, which cut a diameter in the same point, and at equal angles, are equal to one another.

V. Pr. Through a given point, within a given circle, to draw two equal chords, making with one another an angle equal to a given rectilineal angle.

VI. Th. If the diameters of two circles are in the same straight line, and have a common extremity, the two circles shall touch one another.

VII. Pr. Three points being given in the circumference of a circle, and the middle point being equidistant from the other two, to describe two equal circles; which shall touch each other in the middle point, and which shall pass the one through one of the extreme points, and the other through the other extreme point.

VIII. Pr. To draw a tangent to a circle, which shall be parallel to a given finite straight line.

IX. Pr. The diameter of a circle having been produced to a given point, to find in the part produced, a point from which, if a tangent be drawn to the circle, it shall be equal to the segment of the part produced, that is between the given point and the point found.

X. Pr. To describe a circle which shall have a given semi-diameter and its centre in a given straight line, and shall also touch another straight line, inclined at a given angle to the former.

XI. Pr. To describe a circle, the circumference of which shall pass through a given point, and touch a given straight line in another given point.

XII. Pr. To describe a circle, the circumference of which shall pass through a given point, and touch a given circle in another given point; the two points not lying in a tangent to the circle.

XIII. Pr. To describe a circle, which shall touch a given straight line in a given point, and also touch a given circle.

XIV. Pr. To describe two circles, each having a given semi-diameter, which shall touch the same given straight line, both on the same side of it, and shall also touch each other.

XV. Pr. To describe two equal circles, each having its diameter equal to a given straight line, each touching a given circle, and each also passing through a given point without that circle: The given straight line being greater than the shortest distance, between the given point and the circumference of the given circle.

XVI. Pr. To find a point in the diameter, produced, of a given circle, from which, if a tangent be drawn to the circle, it shall be equal to a given straight line.

XVII. Th. If the straight line, drawn from a point in the produced diameter of a circle to the convex circumference be equal to the half of the diameter, the angle at the centre, subtended by the *concave* circumference included between the diameter and the line so drawn, is the triple of the angle, at the centre, subtended by the *convex* circumference included between the same two lines.

XVIII. Pr. Through a given point, either within, or without a given circle, to draw a straight line, so that the part of it within the circle shall be equal to a given finite straight line, which is not greater than the diameter.

XIX. Th. If, from any two points in the circumference of the greater of two given concentric circles, two straight lines be drawn so as to touch the less circle, they shall be equal to one another.

XX. Th. If a quadrilateral rectilineal figure be described about a circle, the angles subtended, at the centre of the circle, by any two opposite sides of the figure, are, together, equal to two right angles.

XXI. Th. If two given straight lines touch a circle, and if any number of other tangents be drawn, all on the same side of the centre, and all terminated by the two given tangents, the angles which they subtend, at the centre of the circle, shall be equal to one another.

XXII. Pr. To draw a tangent to a given circle, such that its segment, contained between the point of contact, and an indefinite straight line, given in position, shall be equal to a given finite straight line.

XXIII. Th. If a straight line touch the interior of two concentric circles, and be terminated both ways by the circumference of the outer circle, it shall be bisected in the point of contact.

XXIV. Th. If a polygon be described about a circle, the straight lines joining the several points of contact will contain a polygon of the same number of angles as the former; and any two adjacent angles of the circumscribed figure shall be, together, the double of that angle, of the inscribed figure, which lies between them.

XXV. Th. If from any given point, in the circumference of a circle, two straight lines be drawn to the extremities of a given chord, the angle which the one makes with any perpendicular to the chord, shall be equal to the angle which the other makes with the diameter of the circle that passes through the given point.

XXVI. Th. The perpendiculars let fall from the three angles of any triangle upon the opposite sides, intersect each other in the same point.

XXVII. Pr. From either of the two given points in which two given circles intersect each other, to draw a chord cutting the one circumference, and meeting the other, such that the part of it, contained between the two circumferences, shall be equal to a given finite straight line.

XXVIII. Th. If two opposite angles of a quadrilateral figure be together equal to two right angles, a circle may be described about it.

XXIX. Th. A circle cannot be described about a rhombus, nor about any other parallelogram which is not rectangular.

XXX. Th. If from any point, in the circumference of a given circle, straight lines be drawn to the three angles of an inscribed equilateral triangle, the greatest of them shall be equal to the aggregate of the two less.

XXXI. Th. The first, third, fifth, &c. angles of any polygon, of an even number of sides, which is inscribed in a given circle, are together equal to the remaining angles of the figure; any angle whatever being assumed as the first.

XXXII. Pr. To make a trapezium, about which a circle may be described, having its four sides respectively equal to four given straight lines, two of which are equal to each other, and any three together greater than the fourth; the two equal sides of the trapezium, also, being opposite to each other.

XXXIII. Pr. Upon a given finite straight line to describe a segment of a circle, which shall be similar to a given segment of another circle.

XXXIV. Th. If upon two opposite sides of an oblong, two similar segments of circles be described, the one of them lying wholly within the oblong, and the other wholly without it, the figure contained by the two remaining sides of the oblong and the two circular arches, shall be equal to the oblong.

XXXV. Th. The arches of a circle that are intercepted between two parallel chords are equal to one another.

XXXVI. Th. If two chords of a given circle intersect each other, the angle of their inclination is equal to the half of the angle at the centre standing upon the aggregate, or the difference, of the arches intercepted between them, accordingly as they meet within, or without the circle.

XXXVII. Th. In equal circles the greater angle stands upon the greater circumference; whether the angles compared be at the centres or the circumferences.

XXXVIII. Th. If from any given point, without a circle, there be drawn two straight lines cutting the circle, then of the circumferences which they intercept, that which is the nearer to the given point is less than the other.

XXXIX. Th. The straight lines joining the extremities of the chords of two equal arches of the same circle, toward the same parts, are parallel to each other.

XL. Th. In equal circles, the greater of two circumferences subtends the greater angle, whether the angles compared be at the centres or the circumferences.

XLI. Pr. If any equilateral rectilineal figure, of an *even* number of sides, be inscribed in a given circle, to find a curvilineal figure that is equal to it, and that is bounded by arches of circles, each of which circles is equal to the given circle.

XLII. Th. In equal circles, the greater chord subtends the greater circumference.

XLIII. Th. If from a given point within a circle, which is not the centre, straight lines be drawn to the circumference, making with each other equal angles, the two, which are nearer to the diameter passing through the given point, shall cut off a greater circumference than the two, which are more remote.

XLIV. Th. In equal circles, the greater circumference has the greater chord.

XLV. Th. The straight line joining any of the angular points of an equilateral polygon inscribed in a circle and the centre, passes through the opposite angular point, or else bisects the opposite side at right angles, accordingly as the figure has an *even*, or an *odd* number of sides.

XLVI. Th. The two straight lines in a circle, which join the extremities of two parallel chords, are equal to each other.

XLVII. Pr. To divide a given circular arch into two parts, so that the aggregate of their chords may be equal to a given straight line, greater than the chord of the whole arch, but not greater than the double of the chord of half the arch.

XLVIII. Pr. To divide a given circular arch into two parts, so that the excess of the chord of the one above the chord of the other, may be equal to a given straight line, less than the chord of the whole arch.

XLIX. Th. If from any point, in the diameter of a semicircle, there be drawn two straight lines to the circumference, one to the bisection of the circumference, the other at right angles to the diameter, the squares upon these two lines are, together, the double of the square upon the semi-diameter.

L. Th. If the chords of two arches of any the same circle cut each other at right angles, the squares of the four segments of the chords, are, together, equal to the square of the diameter.

LI. Pr. To draw a straight line, cutting two concentric circles, the diameter of the greater of which does not exceed the double of the diameter of the less, so that the part of it which lies within the greater circle may be the double of the part which lies within the less.

LII. PR. To draw a straight line which shall touch two given circles.

LIII. TH. If two straight lines, which touch two given circles, the one touching both the circles on the one side of them, the other on the other, be cut by a third tangent, which touches the two circles on contrary sides of them, then, of the segments into which the two first tangents are thus divided, those which are alternate are equal to one another.

LIV. PR. The perimeter, the vertical angle, and the altitude of a triangle being given, to construct the triangle.

LV. TH. If the point, in which two straight lines that are perpendicular to each other meet, be applied to the circumference of a circle so that the straight lines themselves cut the circumference, the centre of the circle is in the bisection of the straight line joining those two intersections.

LVI. TH. If from the extremities of any diameter, of a given circle, perpendiculars be drawn to any chord of the circle, that is not parallel to the diameter, the less perpendicular shall be equal to the segment of the greater contained between the circumference and the chord.

LVII. TH. If from the extremities of any diameter, of a given circle, perpendiculars be drawn to any chord of the circle, they shall meet the chord, or the chord produced, in two points which are equidistant from the centre.

LVIII. TH. If upon either radius, bounding a quadrantal circular arch, as a diameter, a semi-circle be described, any chord of the semi-circle, drawn from the centre of the quadrant, shall be equal to the perpendicular distance of the point, in which the chord produced meets the quadrantal arch, from the other radius.

LIX. TH. If the angle contained by two straight lines, one of which cuts the circle and the other meets it, be equal to the angle in the alternate segment of the circle, the straight line which meets, shall touch the circle.

LX. TH. A straight line touching a circular arch in the bisection of that arch, is parallel to its chord.

LXI. Pr. The base, the vertical angle, and the altitude of a triangle being given, to construct the triangle.

LXII. Pr. To find a point in a given straight line, from which if straight lines be drawn to two given points, on the same side of the given line, they shall contain an angle equal to a given rectilineal angle.

LXIII. Pr. The vertical angle, the base, and the aggregate of the three sides of a triangle being given, to construct the triangle.

LXIV. Pr. The vertical angle, the base, and the excess of the greater of the two remaining sides, of a triangle, above the less, being given, to construct the triangle.

LXV. Pr. From two given points, in the circumference of a circle, to draw two equal chords of that circle, which, produced if necessary, shall make with one another an angle equal to a given rectilineal angle.

LXVI. Pr. In a given parallelogram to inscribe a parallelogram which shall have one of its angles equal to a given angle, and posited in a given point of one of the sides of the given parallelogram.

LXVII. Pr. To produce a given straight line so that the rectangle, under the given straight line, and the part of it produced, shall be equal to a given square.

LXVIII. Th. If through any point in the common chord of two circles, which intersect one another, there be drawn any two other chords, one in each circle, their four extremities shall all lie in the circumference of a circle.

LXIX. Th. If through the given extremity of any diameter of a circle straight lines be drawn to meet an indefinite straight line without the circle, which is perpendicular to the diameter produced, the rectangles contained by the segments of these lines lying between the given point, the point in which each of them cuts the circumference again, and the indefinite line, shall be equal to each other.

LXX. Pr. From the obtuse angle of an obtuse-angled triangle, to draw a straight line to the base, the square of which shall be equal to the rectangle contained by the segments, into which it divides the base.

LXXI. Pr. To make a rectangle which shall be equal to a given square, and shall have its two adjacent sides, together, equal to a given straight line; the side of the given square being less than the half of the given straight line.

LXXII. Th. If from a given point without a circle, two equal straight lines be drawn to the convex circumference, one of which touches the circle, the other shall also touch it.

LXXIII. Pr. To produce a given straight line, so that the rectangle contained by the whole line thus produced, and the part of it produced, shall be equal to a given square.

LXXIV. Th. If, from the bisection of any given arch of a circle, a straight line be drawn cutting the chord of that arch, or the chord produced, and the circumference also of the circle, the rectangle contained by the two parts of the straight line so drawn, the one lying between the point of bisection and the circumference, the other between the point of bisection and the chord, shall be equal to the square of the chord of half the arch.

LXXV. Pr. From the bisection of a given arch of a circle, to draw a straight line, such that the part of it intercepted between the chord, or the chord produced, of the given arch and the circumference, shall be equal to a given straight line.

LXXVI. Pr. Through any given angle of a given equilateral four-sided figure, to draw a straight line terminated by the sides produced, containing the angle opposite to the given angle, which shall be equal to a given straight line.

LXXVII. Th. If two circles cut each other, and from any point, in the straight line produced, which joins their intersections, two tangents be drawn, one to each circle, they shall be equal to one another.

LXXVIII. Th. If two circles cut each other, and if two tangents drawn, one to each circle, from any point without them, be equal, the straight line, joining the intersections of the circles, shall, if it be produced, pass through the common extremity of the equal tangents.

LXXIX. Pr. Two circles being given, neither of which lies within the other, to draw a straight line, such that the tangents, to the two circles, drawn from any point of the line, shall be equal to one another.

LXXX. Pr. To find a point from which if straight lines be drawn to touch three given circles, none of which lies within another, the tangents so drawn shall be equal to one another.

LXXXI. Pr. To divide a given straight line into two parts, so that the square of the one shall be equal to the rectangle contained by the other and a given straight line.

LXXXII. Th. If a given circle be cut by any number of circles, which all pass through the same two given points without the given circle, the straight lines, joining the points of each of these intersections, are either all parallel, or all meet when produced in the same point.

LXXXIII. Th. If a perpendicular be let fall from the right angle, of a right-angled triangle, on the hypotenuse, the rectangle contained by the hypotenuse, and either of the segments, into which it is divided by the perpendicular, is equal to the square of the side adjacent to that segment.

LXXXIV. Th. To draw a tangent to a circle, such, that the part of it intercepted between two straight lines, given in position, but of indefinite length, shall be equal to a given finite straight line:

1st, When the indefinite straight lines both pass through the centre of the circle.

2dly, When they are parallel to one another.

3dly, When they are not parallel, but are equidistant from the centre.

LXXXV. Th. If from the intersection of any two tangents to a circle, any straight line be drawn, cutting the chord which joins the two points of contact and again meeting the circumference, it shall be divided by the circumference and the chord into three segments, such, that the rectangle contained by the whole line and the middle part, shall be equal to the rectangle contained by the extreme parts.

LXXXVI. Pr. To make a rectangle which shall be equal to a given square, and have the difference between its two adjacent sides equal to a given straight line.

LXXXVII. Pr. From a given point without a circle, to draw a straight line cutting the circle, so that the rectangle contained by the part of it without, and the part within, the circle, shall be equal to a given square.

LXXXVIII. Pr. To describe a circle which shall touch a given straight line, and pass through two given points, both on the same side of the given line, and in the same plane with it.

LXXXIX. Pr. To describe a circle which shall have its centre in a given straight line, which shall pass through a given point, and shall, also, touch another given straight line.

XC. Pr. To describe a circle which shall touch two given straight lines, and pass through a given point between them.

XCI. Pr. To describe a circle which shall touch two given straight lines, and also touch a given circle, which does not lie wholly without the two given straight lines.

XCII. Pr. To describe a circle which shall touch both a given circle, and a given straight line, and which shall, also, pass, first, through a given point without the given circle; and, secondly, through a given point within the circle.

XCIII. Pr. In a straight line of indefinite length, but given in position, which cuts a given circle, to find a point, from which if a straight line be drawn to touch the circle, it shall be equal to a given finite straight line.

XCIV. Pr. To describe a circle that shall touch a given straight line, and that shall also touch two given circles.

XCV. PR. To describe a circle which shall touch a given circle, and pass through two given points, either both without the circle, or both within it.

XCVI. PR. To find a point in a straight line, given in position, from which if two straight lines be drawn to two given points, without the given line, they shall have, first, their difference, and, secondly, their aggregate, equal to a given finite straight line.

XCVII. PR. The base and the altitude of a triangle being given, together with the aggregate, or the difference, of the two remaining sides, to construct the triangle.

XCVIII. PR. Three points being given, to find a fourth, from which if straight lines be drawn to the other three, two of them shall be equal, and the difference between either of these and the third shall be equal to a given straight line.

XCIX. PR. To describe a circle that shall touch three given circles, of which two are equal to one another.

C. PR. To find a point, in the circumference of a given circle, from which if two straight lines be drawn to two given points, without the circle, the chord joining the intersections of the lines so drawn and the circumference, shall be parallel to the straight line joining the two given points.

BOOK IV.

PROP. I. TH. If an equilateral triangle be described about a given circle, the straight lines joining the points of contact shall contain another equilateral triangle; and the side of the circumscribed triangle is the double of the side of the inscribed triangle so contained.

II. TH. If a triangle be described about a given circle, the rectangle contained by the perimeter of the triangle and the semi-diameter of the circle shall be double of the triangle.

III. Pr. Three straight lines being given, which, when produced, do not all three meet in the same point, and of which the middle line is not parallel to either of the others, to describe a circle which shall touch each of them.

IV. Th. The three straight lines, which bisect the three angles of a triangle, meet in the same point.

V. Th. If a circle be inscribed in a right-angled triangle, the excess of the two sides, containing the right angle, above the third side, is equal to the diameter of the inscribed circle.

VI. Th. The straight line bisecting any angle of a triangle, inscribed in a given circle, cuts the circumference, in a point which is equidistant from the extremities of the side opposite to the bisected angle, and from the centre of a circle inscribed in the triangle.

VII. Pr. In a given circle, to inscribe three equal circles, touching each other and the given circle.

VIII. Pr. To inscribe three circles in an equilateral triangle, touching each other, and each of them touching two of the three sides of the triangle.

IX. Th. The square, inscribed in a circle, is equal to the half of the square upon its diameter.

X. Pr. In a given circle, to inscribe a rectangle equal to a given rectilineal figure, not exceeding the half of the square upon the diameter.

XI. Th. If from any point, in the circumference of a given circle, straight lines be drawn to the four angular points of an inscribed square, the aggregate of the squares of the four lines, so drawn, shall be the double of the square of the diameter.

XII. Pr. In a given circle, to inscribe four circles equal to each other, and in mutual contact with each other and the given circle.

XIII. Pr. To inscribe a circle in a given trapezium, of which two opposite sides are, together, equal to the other two sides taken together.

XIV. Pr. Upon a given finite straight line, to describe an equilateral and equiangular decagon.

XV. Pr. Upon a given finite straight line, to describe an equilateral and equiangular pentagon.

XVI. Th. The angle of a regular pentagon exceeds a right angle by one-fifth part of a right angle; and is three times as great as the angle contained by any two sides of the figure, which are not adjacent to each other, produced so as to meet.

XVII. Th. The square of the side of a regular pentagon, inscribed in a given circle, is equal to the square of the side of a regular decagon, together with the square of the side of the regular hexagon, both inscribed in that given circle.

XVIII. Pr. Upon a given finite straight line, to describe an equilateral and equiangular hexagon.

XIX. Pr. A circle being given, to describe six other circles, each of them equal to it, and in contact with each other and with the given circle.

XX. Pr. In a given circle to inscribe six circles equal to one another, touching, each of them, the given circle, and touching, also, one another.

Book V.

Prop. I. Th. If the first of four proportional magnitudes be greater than the second, the third is also greater than the fourth; if equal, equal; and if less, less.

II. Th. If four magnitudes are proportionals, they are proportionals also when taken inversely.

III. Th. If the first of four magnitudes be the same multiple of the second, or the same part of it, that the third is of the fourth, the first is to the second, as the third is to the fourth.

IV. TH. If the first of four proportional magnitudes be a multiple, or a part, of the second, the third is the same multiple, or the same part, of the fourth.

V. TH. If any number of equal ratios be each greater than a given ratio, the ratio of the sum of their antecedents to the sum of their consequents, shall be greater than that given ratio.

VI. TH. If the first of four magnitudes have a greater ratio to the second than the third has to the fourth, the second shall have to the first a less ratio than the fourth has to the third.

VII. TH. If the first of four magnitudes, of the same kind, have a greater ratio to the second than the third has to the fourth, the first shall have to the third a greater ratio than the second has to the fourth.

VIII. TH. If four magnitudes of the same kind be proportionals, and if the first of them be the greatest, the fourth shall be the least; but if the first of them be the least, the fourth shall be the greatest.

IX. TH. If the first, together with the second, of four magnitudes, have a greater ratio to the second, than the third, together with the fourth, has to the fourth, the first shall have a greater ratio to the second than the third has to the fourth.

X. TH. If the first of four magnitudes have a greater ratio to the second than the third has to the fourth, the first, together with the second, shall have to the second, a greater ratio than the third, together with the fourth, has to the fourth.

XI. TH. If the first term of a ratio be less than the second, the ratio shall be increased by adding the same quantity to both terms; but if the first term be greater than the second, the ratio shall be diminished by adding the same quantity to both.

XII. TH. If the first of four magnitudes, of the same kind, have a greater ratio to the second than the third has to the fourth, the first, together with the third, shall have to the second, together with the fourth, a greater ratio than the third has to the fourth, and a less ratio than the first has to the second.

XIII. TH. If the first, together with the second, have to the second, a greater ratio than the third, together with the fourth, has to the fourth, then shall the first, together with the second, have to the first, a less ratio than the third, together with the fourth, has to the third.

XIV. TH. If the first, together with the second, have to the third, together with the fourth, a greater ratio than the first has to the third, then shall the second have to the fourth a greater ratio, than the first, together with the second, has to the third, together with the fourth.

XV. TH. If four magnitudes be proportionals, they are also proportionals by conversion: that is, the first is to its excess above the second, as the third to its excess above the fourth.

XVI. TH. If the first of four magnitudes of the same kind, have a greater ratio to the second, than the third has to the fourth, and if the aggregate of the first and second be greater than the aggregate of the third and fourth, the first shall be greater than the third.

XVII. TH. If any number of magnitudes be continual proportionals, their differences shall, also, be proportionals.

XVIII. TH. If there be three magnitudes, and other three, and if the first have a greater ratio to the second, in the former set, than the first has to the second, in the latter; and if, also, the second have to the third, in the former set, a greater ratio than the second has to the third, in the latter; then shall the first have a greater ratio to the third, in the former set, than the first has to the third, in the latter.

XIX. TH. If there be three magnitudes, and other three, and if the first have to the second, in the former set, a greater ratio than the second has to the third, in the latter; and if, also, the second have to the third, in the former set, a greater ratio than the first has to the second, in the latter; then shall the first have to the third, in the former set, a greater ratio, than the first has to the third, in the latter.

XX. Th. If three magnitudes be proportionals, the two extremes are, together, greater than the double of the mean.

XXI. Th. If there be two sets of magnitudes, the one geometric, and the other arithmetic, proportionals, and if the two first magnitudes be the same in both, any other magnitude in the former set, shall be greater than the corresponding magnitude in the latter.

XXII. Th. If there be two series of magnitudes, the one arithmetically proportional, the other geometrically proportional, but each having the same magnitude for its first term, and if the last term of the arithmetic series be not less than the last term of the geometric series, any other term of the former series shall be greater than the corresponding term in the latter.

Book VI.

Prop. I. Th. If the bases of four rectangles be proportionals, and their altitudes be also proportionals, the rectangles themselves shall likewise be proportionals.

II. Th. If the outward angle of a triangle, made by producing one of its sides, be divided into two equal angles, by a straight line which also cuts the base produced, the segments between the dividing line and the extremities of the base have the same ratio, which the other sides of the triangle have to one another: And if the segments of the base, produced, have the same ratio which the other sides of the triangle have, the straight line, drawn from the vertex to the point of section, divides the outward angle of the triangle into two equal angles.

III. Th. Either of the equal sides of an isosceles triangle, is a mean proportional between the base, and the half of the segment of the base, produced if necessary, which is cut off by a straight line drawn from the vertex at right angles to the equal side.

IV. Th. The diameter of a circle is a mean proportional between the sides of an equilateral triangle and hexagon described about the circle.

V. Th. Equiangular parallelograms have to one another the same ratio as the rectangles contained by the sides about equal angles in each.

VI. Th. The straight lines, drawn from the bisections of the three sides of a triangle to the opposite angles, meet in the same point.

VII. Pr. To find, within a given rectilineal angle, first, the *locus* of all the points, from each of which, if two straight lines be drawn, to the lines containing the given angle, so as always to be parallel to two straight lines given in position, they shall be to one another in a given ratio: And secondly, to find the *locus* of all the points, from each of which if two straight lines be drawn in like manner, they shall cut off from two given parts of the straight lines containing the given angle, segments that shall be to one another in a given ratio.

VIII. Th. If a circle be touched, in the same point, both externally and internally, by two other circles, and through the point of contact two straight lines be drawn, the parts of them intercepted between the circumference of the given circle, and that of the circle which touches it internally, shall have to one another the same ratio as the parts which are chords of the other circle.

IX. Pr. From the centre of a given circle, to draw a straight line to meet a given tangent to the circle, so that the segment of the line between the circle and the tangent shall be any required part of the tangent.

X. Pr. From a given triangle to cut off a rhombus; the base of the rhombus being part of the base of the triangle, and having its extremity in a given point of that base.

XI. Th. If two triangles have one angle of the one, equal to one angle of the other, and also another angle of the one, together with another angle of the other, equal to two right angles, the sides about the two remaining angles shall be proportionals.

XII. Th. If, from the extremities of the base of a given triangle, there be drawn two straight lines, both on the same side of the base, and each equal to the adjacent side, and making with that side an angle equal to the vertical angle of the triangle, then the straight lines which join the extremities of the lines so drawn, and the further extremities of the base, shall cut off, from the sides, equal segments towards the vertex; and each of those segments shall be a mean proportional between the other segments, that are towards the base.

XIII. Th. If at the extremities of the hypotenuse of a right-angled triangle two straight lines be drawn, on the same side of the hypotenuse as the right angle, each equal to, and each perpendicular to, the adjacent side, the two straight lines joining each of their extremities and the further extremity of the hypotenuse, shall cut each other in the same point of the perpendicular drawn to the hypotenuse from the right angle.

XIV. Th. The semi-circumference of a circle having been divided into any number of equal parts, and chords having been drawn, from either extremity of the diameter, to the several points of division, the first chord has to the second, the same ratio which the second has to the aggregate of the first and third; or the same ratio which any other chord has to the aggregate of the two chords that are next to it.

XV. Pr. From a given point, either within or without a given rectilineal angle, to draw a straight line cutting off from the lines which contain the angle, segments, towards the summit of the angle, which shall be to one another in a given ratio.

XVI. Pr. To draw through a given point a straight line cutting the lines which contain a given rectilineal angle, so that the segment of it, between those lines, shall be divided by the straight line that bisects the given angle, into two parts, which are to one another in a given ratio.

XVII. Th. If two trapeziums have an angle of the one equal to an angle of the other, and if, also, the sides of the two figures, about each of their angles, be proportionals, the remaining angles of the one shall be equal to the remaining angles of the other.

XVIII. Th. If two straight lines touch a circle at opposite extremities of its diameter, any other tangent of the circle, terminated by them, is so divided in its point of contact, that the radius of the circle is a mean proportional between its segments.

XIX. Th. If two given circles touch each other, and also touch a given straight line, the part of the line between the points of contact, is a mean proportional between the diameters of the circles.

XX. Pr. The two straight lines being given, which are the two first of a series of proportionals, to find the rest; and, if the series decrease, to find a line which shall be greater than the aggregate of any number, whatever, of its terms, but to which the aggregate may approximate indefinitely.

XXI. Pr. To describe a square which shall have a given ratio to a given rectilineal figure.

XXII. Pr. To divide a given finite straight line into two parts, the squares of which shall be to one another in a given ratio.

XXIII. Pr. To find two points situated in two adjacent sides of a given oblong, at equal distances from two opposite angles, from which, if two straight lines be drawn parallel to the sides of the figure, they shall cut off from it any part required.

XXIV. Pr. Within a given oblong, to describe another oblong which shall be any required part of it, and shall have its four sides all equally distant from the four sides of the given rectangle.

XXV. Pr. The base, the vertical angle, and the ratio of the two sides of a triangle being given, to construct it.

XXVI. Pr. A given finite straight line being divided into any two given parts, to divide it again, so that the rectangle contained by the two former given parts shall have a given ratio to the rectangle contained by the two latter parts.

XXVII. Pr. To draw a straight line to touch a given arch of a circle, so that being terminated by the semi-diameters, produced, which bound the arch, it shall be divided by the point of contact, into two parts that are to one another in a given ratio.

XXVIII. Pr. Two points being given, one in each of two parallel straight lines, and a third point being also given, without them, to draw, from that third point, a straight line so to cut the parallels, as that the segments of the parallels, between it and the two first points, shall be to one another in a given ratio.

XXIX. Pr. To find a point within a given triangle, from which if three straight lines be drawn to the three angles of the triangle, it shall thereby be divided into three parts that are each to each in given ratios.

XXX. Pr. To divide a given circular arch into two parts, so that the chords of those parts shall be to each other in a given ratio.

XXXI. Pr. To inscribe a square in a given trapezium, which has the two sides about any angle equal to one another, and the two sides about the opposite angle also equal to one another.

XXXII. Pr. To inscribe a square in a given trapezium.

XXXIII. Pr. To determine the locus of the summits of all the triangles which can be described on a given base, so that each of them shall have its two sides in a given ratio.

XXXIV. Pr. The base, the perpendicular distance of the vertex from the base, and the ratio of the two sides of a triangle being given, to construct it.

XXXV. Pr. The segments into which the perpendicular, drawn from the vertex to the base of a triangle, divides the base, and the ratio of the two remaining sides being given, to construct the triangle.

XXXVI. Pr. To find a point, from which if three straight lines be drawn to three given points, they shall be each to each in given ratios.

XXXVII. Pr. A straight line being divided into three given parts, to find a point without it, at which the three parts shall subtend equal angles.

XXXVIII. Pr. To find a point in a given line, from which, if two straight lines be drawn to two given points, both on the same side of the given line, they shall be to each other in a given ratio.

XXXIX. Pr. In a given parallelogram to inscribe a parallelogram that shall have its two adjacent sides in a given ratio to one another, and that shall be the half of the given parallelogram.

XL. Pr. From a given point, either within or without a given rectilineal angle, to draw a straight line cutting the two lines which contain the angle, so that the distances of the two intersections from the given point, shall be to one another in a given ratio.

XLI. Pr. To find, between two given parallel straight lines, the locus of all the points, from each of which if two straight lines be drawn to the two given parallels, so as always to make with them, towards the same parts, given angles, they shall be to one another in a given ratio.

XLII. Pr. To divide a given straight line into two parts, such, that the rectangle contained by the whole line and one of its parts, shall have a given ratio to the square of the other part.

XLIII. Pr. One given circle lying within another, to find a point from which, if two tangents be drawn, one to each of the given circles, they shall be to each other in a given ratio.

XLIV. PR. From a given point, to draw a straight line to cut a given circle, so that the distances of the two intersections from the given point, shall be to each other in a given ratio.

XLV. PR. Two given circles lying wholly without one another, through a given point, which is between the two circles, and which is posited in the straight line joining their centres, to draw a straight line that shall be terminated by the convex circumferences, and divided, by the given point, into two parts, that are to one another in a given ratio.

XLVI. PR. To find a point, from which if three straight lines be drawn to meet as many given straight lines, which cut one another, so as to make, each with the line on which it falls, an angle equal to a given angle, the lines so drawn shall be, each to each, in given ratios.

XLVII. PR. To make an isosceles triangle, which shall be equal to a scalene triangle, and shall also have an equal vertical angle with it.

XLVIII. TH. If a straight line, drawn from the vertex of an isosceles triangle cutting the base, be produced to meet the circumference of a circle described about the triangle, the rectangle contained by the whole line so produced, and the part of it between the vertex and the base, shall be equal to the square of either of the equal sides of the triangle.

XLIX. TH. If from a given point, without a circle, two straight lines be drawn to the concave circumference, they shall be reciprocally proportional to the parts of them between the given point and the convex circumference.

L. PR. To divide a given finite straight line into two parts such that another given straight line, not greater than the half of the former, shall be a mean proportional between them.

LI. PR. Of four straight lines which are continual proportionals, the two extremes being given, and also a line which is equal to the difference of the other two, to find those two lines.

LII. Pr. To make a triangle, which shall have its two sides equal to two given straight lines, each to each, and shall have its base equal to the perpendicular distance of the vertex from the base.

LIII. Th. If from any point in the diameter, or the diameter produced, of a given parallelogram perpendiculars be let fall on the two adjacent sides, produced, if necessary, which meet the diameter, the perpendiculars shall be reciprocally proportional to the sides on which they fall.

LIV. Pr. From a given point, in the base of a scalene triangle, to draw a straight line, which shall cut off equal segments from the two remaining sides, the less of those sides having been produced.

LV. Th. If an angle of a triangle be bisected by a straight line, which also cuts the base, the rectangle, contained by the sides of the triangle, is equal to the rectangle contained by the segments of the base, together with the square of the straight line bisecting the angle.

LVI. Th. If from any angle of a triangle a straight line be drawn perpendicular to the base, the rectangle contained by the sides of the triangle, is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

LVII. Th. The rectangle contained by the diagonals of a quadrilateral rectilineal figure, inscribed in a circle, is equal to both the rectangles contained by its opposite sides.

LVIII. Th. If, from the centre of the circle, described about a given triangle, perpendiculars be drawn to the three sides, their aggregate shall be equal to the radius of the circumscribed circle, together with the radius of the circle inscribed in the given triangle.

LIX. Pr. To find a point, from which if three straight lines be drawn to three given points, their differences shall be severally equal to three given straight lines; the difference of any two of the straight lines to be drawn, not being greater than the distance of the two points to which they are to be drawn.

LX. Pr. To describe a circle, which shall pass through a given point, and touch two given circles.

LXI. Pr. To describe a circle that shall touch three given circles.

LXII. Pr. Upon a given finite straight line, to describe an equilateral and equiangular figure, having the number of its sides equal to four, eight, sixteen, &c.; or to three, six, twelve, &c.; or to five, ten, twenty, &c.; or to fifteen, thirty, sixty, &c. sides.

LXIII. Th. Similar triangles, and similar polygons, are to one another as any rectilineal figure described upon any side of the one, is to a similar rectilineal figure similarly described upon the homologous side of the other.

LXIV. Pr. To cut off from a given triangle any part required, by a straight line drawn parallel to a given straight line.

LXV. Pr. To describe a polygon, similar to a given polygon, and having a given ratio to it.

LXVI. Th. Any regular polygon, inscribed in a circle, is a mean proportional between the inscribed and circumscribed regular polygons of half the number of sides.

LXVII. Th. If from two points similarly situated, one in each of any two homologous sides of two similar polygons, two straight lines be drawn making equal angles with those sides, they shall cut off from the polygons two similar figures; and the one shall be the same part of the one polygon, that the other is of the other.

LXVIII. Th. If any two chords of a circle intersect each other, the straight lines joining their extremities shall cut off equal segments from the chord which passes through the common intersection of the two former chords and is there bisected.

LXIX. Pr. Two similar rectilineal figures being given, to find a third figure also similar to them and a mean proportional between them.

LXX. PR. If two sides of a trapezium be parallel; and a straight line be drawn cutting them, and meeting also the other two sides, (any of the sides being produced, if necessary) the two rectangles contained by the respective segments of the parallel sides, have to each other the same ratio, as the two rectangles contained by the segments into which the line, so drawn, is severally divided by each of the two parallels.

LXXI. PR. To cut off from a given parallelogram a similar parallelogram, which shall be any required part of it.

LXXII. TH. A given straight line being cut in extreme and mean ratio, if from the greater segment the less be taken, the greater segment also will thus be cut in extreme and mean ratio; and if a straight line, equal to the greater segment, be added to the given line, the line which is made up of the given line and this segment, is also cut in extreme and mean ratio.

LXXIII. PR. Upon a given straight line, as an hypotenuse, to describe a right-angled triangle, which shall have its three sides continual proportionals.

LXXIV. PR. The perimeter being given of a right-angled triangle, having its three sides proportionals, to construct the triangle.

LXXV. TH. The semi-diameter of a given circle having been divided in extreme and mean ratio, the greater segment shall be equal to the side of an equilateral and equiangular decagon inscribed in the circle.

LXXVI. TH. Any rectangle is the half of the rectangle contained by the diameters of the squares of its two sides.

LXXVII. PR. Through a given point, to draw a straight line, cutting two given straight lines, which meet one another, so that the triangle contained by the segment of that line and the segments which it cuts off from the given lines, shall be equal to a given rectilineal figure.

LXXVIII. PR. Through a given point to draw a straight line, so as to cut off from two straight lines, that meet one another, two segments, toward their point of concourse, which shall contain a rectangle equal to a given square.

LXXIX. Th. In different circles the semi-diameters which bound equal sectors contain angles reciprocally proportional to their circles; and conversely.

LXXX. Pr. A given space being bounded by two arches of circles, subtending, at their respective centres, angles reciprocally proportional to the circles, to find a square that shall be equal to it.

LXXXI. Pr. To trisect a given circle, by dividing it into three equal sectors.

LXXXII. Th. If, from the greater of two unequal sides of a given triangle, be cut off a part equal to the less, that segment shall have to the remaining segment, a ratio greater than the ratio which the angle adjacent to the remaining segment, has to the angle adjacent to the segment first cut off.

LXXXIII. Th. The greater of any two unequal arches, of a given circle, has a greater ratio to the less arch, than the chord of the greater has to the chord of the less.

LXXXIV. Th. The greater angle, at the base of a scalene triangle, has a greater ratio to the less angle, than the greater side has to the less side.

LXXXV. Pr. To divide a given circle into any required number of equal parts, by the circumferences of circles described within it, about its centre.

LXXXVI. Pr. To find a circle, which shall be equal to the excess of the greater of two given circles above the less.

LXXXVII. Pr. To find a circle to which a given circle shall have the same ratio, as that which one given straight line has to another.

LXXXVIII. Th. If, in any given circle, two chords cut each other at right angles, the four circles described upon their segments, as diameters, shall, together, be equal to the given circle.

LXXXIX. Th. A circle is equal to the half of the rectangle contained by its semi-diameter and by a straight line which is equal to its circumference.

XC. Th. A circle is a mean proportional between any regular polygon, described about it, and a similar polygon, the perimeter of which is equal to the circumference of the circle.

PART II.

Book I.

Prop. I. Th. If two chords of a circle cut each other at right angles, either pair of opposite arches, intercepted between them, is equal to the semi-circumference of the circle.

II. Th. If three equal chords of a circle cut one another in the same point, within the circle, that point is the centre, and the chords are diameters of the circle.

III. Pr. To find the locus of the extremities of all the straight lines, that can be drawn from the circumference of a given circle, towards the same parts, each of them parallel and equal to a given finite straight line.

IV. Th. If a chord of the greater of two concentric circles, cut the less of them, its segments, between the two circumferences, shall be equal to one another.

V. Th. If any number of chords of the greater of two concentric circles, cut the less, the rectangles contained by the two segments, into which each of them is divided by the circumference of the less circle, shall be equal to one another.

VI. Th. Of parallel straight lines, terminated by the concave circumference of the greater of two concentric circles, and the convex circumference of the less, the least is that, which, produced, passes through the centre; and of the rest, that, which is nearer to the least, is always less than the more remote.

VII. Pr. To draw, in a given circle, a chord, which shall be equal and parallel to a given finite straight line, not exceeding the diameter of the circle.

VIII. Pr. To place between the concave circumference of the greater of two given concentric circles, and the convex circumference of the less, a straight line which shall be equal and parallel to a given finite straight line, which is not greater than a tangent drawn from any point of the less to meet the greater circumference, and not less than half the difference of the diameters of the two circles.

IX. Pr. From a given point, without a given circle, to draw a straight line cutting the circle, so that its segment, which is within the circle, shall be equal to a given straight line, not greater than a diameter of the circle.

X. Th. If from the extremities of any chord of a circle, straight lines be drawn at right angles to the chord, they shall meet any diameter of the circle in points, that are equidistant from the centre.

XI. Th. If two given finite straight lines, which are neither in the same straight line, nor parallel to each other, be produced so as to meet, and if the rectangle contained by the one line so produced, and the part of it produced, be equal to the rectangle contained by the other line pruduced, and the part of it produced, the extremities of the two given lines, shall be in the circumference of a circle.

XII. Th. If, in any two given circles, that touch one another, there be drawn any two parallel diameters, an extremity of each diameter, and the point of contact, shall lie in the same straight line.

XIII. Th. If two circles touch each other internally, any chord of the greater circle, which touches the less, shall be divided, by the point of its contact, into segments that subtend equal angles at the point, in which the two circles touch one another.

XIV. Th. If two equal chords, in a given circle, cut one another, the segments of the one shall be equal to the segments of the other, each to each.

XV. Pr. To draw two tangents to a given circle, which shall make with one another an angle, equal to a given angle; and shall meet in a given diameter produced.

XVI. Pr. In the straight line, joining the centres of two given circles, that lie wholly without one another, to find a point, from which if two tangents be drawn to the two circles, they shall make equal angles with that line.

XVII. Th. If two arches of unequal circles have a common chord, the arch, which belongs to the greater circle, shall have the less sagitta.

XVIII. Th. If the arch of a segment of a circle be divided into any two parts, the chord of the whole arch shall exceed the chord of the one part, by the double of the segment, which is cut off from the chord of the whole arch, by a straight line drawn at right angles to it, from the bisection of the other part.

XIX. Th. If the curve of a semi-circle be subtended by any three chords, the square of the diameter shall exceed the squares of the three chords, by twice the rectangle contained by the middle chord, and the part of it produced, intercepted between its extremity, and the perpendicular drawn to it from the extremity of the diameter.

XX. Pr. If any number of quadrilateral rectilineal figures be inscribed in a given circle, having the same chord for their common base, and a diameter of the circle, for the side opposite to the base, to find the locus of the intersections of their two diagonals in each of them.

XXI. Pr. Through two given points in a diameter of a given circle, both on the same side of the centre, to draw two parallel chords, which shall intercept the two greatest arches.

XXII. Th. If a quadrilateral rectilineal figure, having two of its sides parallel, be described about a circle, a straight line drawn through the centre of the circle, parallel to either of the two parallel sides, and terminated by the other two sides shall be equal to a fourth part of the perimeter of the figure.

XXIII. Th. The circumference of a circle is a curve of uniform curvature.

XXIV. Th. If, from any of the angles of any given rectangle, a straight line can be drawn cutting the two straight lines that contain the opposite angle, so that its segment, included between those two lines, shall be the double of the rectangle's diameter, then any given plane rectilineal angle can be trisected.

XXV. Th. If from the hypotenuse of a right-angled triangle, a segment be cut off equal and adjacent to either of the other two sides, and if from the right angle a perpendicular be drawn to the hypotenuse, the straight line joining the right angle and the end of the segment, shall bisect the angle contained by the perpendicular and the third side of the triangle.

XXVI. Pr. From a given point, in the circumference of a given circle, to draw a straight line, which shall touch the circle, by the help only of the *parallel ruler*.

XXVII. Th. The least square, which can be inscribed in a given square, is that which is the half of the given square.

XXVIII. Th. If an oblong be inscribed in a square, each of its sides cuts off equal segments from the sides of the square.

XXIX. Th. An oblong inscribed in a given square, is less than the least square inscribed in it.

Book II.

Prop. I. Pr. To divide a given finite straight line into two unequal parts, of which the less shall be a mean proportional between the greater, and the part which lies between the point of section, and the middle point of the given line.

II. Pr. A given straight line being divided into any two parts, to divide it again, so that the rectangle, contained by the whole line, and the part between the two points of section, shall be equal to the square of the part between the second point of section and the extremity, nearest to it, of the given line.

III. Pr. A given straight line being divided into any two parts, to divide it again, so that the rectangle, contained by the whole line, and the parts between the points of section, shall be equal to the rectangle, contained by the two parts, into which the given line is divided by the second point of section.

IV. Pr. A straight line having been divided into two equal parts, to divide it again, so that the rectangle, contained by the two unequal parts, shall be to the square of the line between the two points of section, in a given ratio.

V. Pr. Two points C, and D, being given, in a finite straight line AB, which are neither of them one of its extremities, to find an intermediate point E, such that the rectangle $\overline{AE} \times \overline{EC}$, shall be to the rectangle $\overline{BE} \times \overline{ED}$, in a given ratio.

VI. Pr. To produce a given straight line, so that the square of the line, which is made up of the whole and the part produced, together with the square of the part produced, shall be in a given ratio to a given square.

VII. Pr. To produce a given straight line, so that the square of the given line, together with twice the rectangle contained by the given line and the part produced, shall be to the square of the part produced, in a given ratio.

VIII. Pr. To divide a given finite straight line into two parts, so that the rectangle, contained by the whole line, and the difference of those two parts, shall be in a given ratio to the square of the less of the two parts.

IX. Pr. To divide a given finite straight line into two parts, the less of which shall be in a given ratio to their difference.

X. Pr. To divide a given finite straight line into three parts, that shall be continual proportionals, and the square of the greatest of which shall be equal to the squares of the other two.

XI. Pr. To divide a given finite straight line into four parts, which shall be proportionals, according to a given ratio, and shall, also, be such, that, if the least be added to each of the three least, the greatest, and the three aggregates, shall be proportionals.

XII. Pr. To divide a given finite straight line into two parts, such that the aggregate of their squares shall be in a given ratio to a given square.

XIII. Pr. To divide a given finite straight line into two parts, such that the difference of their squares shall be in a given ratio to a given square.

XIV. Pr. To divide a given finite straight line into four proportionals, so that the aggregate of the extremes shall be in a given ratio to the aggregate of the means.

XV. Pr. To divide a given finite straight line into four proportionals, so that the difference of the extremes shall be to the difference of the means, in a given ratio.

XVI. Pr. Three points, in a given straight line being given, to find, in that line, a fourth point, between the second and third, such that the rectangle contained by the distance between the first and fourth, and the distance between the second and fourth, shall be equal to the square of the distance between the third and the fourth points.

XVII. Pr. Four points being given in the same straight line, to find, in that line, a fifth point between the second and third, such that the square of the distance between the first and the second, together with the square of the distance between the first and the fifth, shall be to the square of the distance between the third and the fourth, together with the square of the distance between the fourth and the fifth, in a given ratio.

XVIII. Pr. Two points being given, in a given finite straight line, to find a third intermediate point, such that the rectangle contained by the two parts into which the line is divided by the first given point, together with the square of the distance between that point and the third point, shall be in a given ratio to the rectangle contained by the parts, into which the line is divided by the second given point, together with the square of the distance between that point and the third point.

XIX. Pr. The first of three continual proportionals being given, and also the aggregate of the other two, to find those other two.

XX. Pr. To find two squares, the difference between which shall be in a given ratio to the difference between the less of them, and a given square.

XXI. Th. Two unequal magnitudes being given, an intermediate magnitude may be found, which shall be commensurable with a third given magnitude, of the same kind.

XXII. Th. Two incommensurable magnitudes being given, a third magnitude may be found, commensurable with one of them, and the difference between which and the other, shall be less than any assigned magnitude, the magnitudes being all of the same kind.

XXIII. Pr. Two unequal finite straight lines being given, to find another intermediate straight line which shall be incommensurable with a third given finite straight line.

Book III.

Prop. I. Th. The two sides of a triangle are to one another, reciprocally, as the perpendiculars drawn to them from the bisection of the base.

II. Th. If perpendiculars be drawn from the three angles of an acute-angled triangle, to the opposite sides, the rectangles contained by the several sides and their corresponding segments, shall be equal: Also, the rectangle contained by the two segments of any one of the perpendiculars, shall be equal to the rectangle contained by the segments of any other of them.

III. Pr. From any of the angles of an equilateral parallelogram to draw a straight line, cutting the two lines, which contain the opposite angle, so that its segment, included between those lines, may be equal to a given finite straight line.

IV. Th. If from any two points, in the semi-circumference of a given circle, straight lines be drawn to the extremities of the diameter, which subtends the semi-circumference, the rectangle contained by the two, that are terminated in one extremity of the diameter, together with the rectangle contained by the other two, shall be less than the square of the diameter.

V. Th. If from the bisection of any given arch of a circle, two straight lines be drawn, cutting the chord of the arch and the circumference, the four points of intersection shall also lie in the circumference of a circle.

VI. Th. If a given arch of a circle be divided into three equal parts, their chords shall, together, exceed the chord of the whole arch, by a straight line, which is to the chord of one of the equal parts as the square of that chord is to the square of the semi-diameter.

VII. Th. The side of an equilateral polygon, inscribed in a circle, is a mean proportional between the side of a similar polygon described about it, and the half of the side of an equilateral polygon of half the number of sides inscribed in the circle.

VIII. Th. If two tangents be drawn to a circle, from any given point without it, and if, from either of the two points of contact, there be drawn a perpendicular to that diameter, which passes through the other point of contact, the perpendicular shall be bisected by the straight line joining the given point and the end of the diameter.

IX. Th. The rectangle contained under the tangent and the sine, of any arch of a circle, is greater than the square of the chord of the arch.

X. Th. If the diameter, passing through the extremity of a given arch of a circle, be produced, so that the part produced shall be equal to the versed sine of the given arch, and if, from the end of the part produced, a tangent be drawn to the circle, its square shall exceed the square of the versed sine, by the square of the chord of the arch.

XI. Pr. Three sides of a quadrilateral rectilineal figure, of which two are equal to one another, being given, to construct it, so that the fourth side shall be the diameter of a circle described about the figure.

XII. Pr. To describe the circumference of a circle, which shall pass through two given points, within a given circle, and shall cut the circumference of that circle in two points, the distance of which shall be equal to a given straight line, not greater than the diameter of the given circle.

XIII. Pr. In the given straight line joining the centres of two circles, that lie wholly without one another, to find a point, such that if tangents be drawn from it to the two circles, the perpendiculars drawn from the points of contact, to that given line, shall cut off from the diameters two segments, which shall be to each other, as the diameters are to each other.

XIV. Pr. To draw a tangent to a given circle, to meet a given diameter produced, so that the tangent shall have a given ratio of inequality to the part produced of the diameter, which is terminated by the tangent.

XV. Th. If two unequal chords of a circle be terminated in the same given point, and if, from the other extremity of either of them, there be drawn a perpendicular to the diameter, which terminates in the given point, that chord shall be a mean proportional between the other given chord, and the segment cut off from it, by the perpendicular.

XVI. TH. The square of the side of an equilateral and equiangular octagon, inscribed in a given circle, and the square of the diameter of the circle, have no common measure.

XVII. PR. In a given circle, to draw a chord parallel to a given straight line, so as that it shall be divided, by another given chord, into two parts, which shall be to one another in a given ratio of inequality.

XVIII. PR. To find a point, in the circumference of a given semi-circle, from which if a perpendicular be drawn to the diameter, it shall be divided, by the arch of another circle, which has the semi-circle's diameter for its chord, similarly to a given divided straight line.

XIX. PR. To divide the quadrant of the circumference of a circle, into two arches, such that the aggregate of their versed sine shall be equal to the right sine of the less.

XX. PR. To divide the quadrant of the circumference of a circle into two arches, such that the aggregate of the tangent, and the secant of the lesser arch, shall be equal to the diameter of the circle.

XXI. PR. In the given straight line joining the centres of two given circles, that lie wholly without each other, to find a point, such that the two tangents, drawn from it to the two given circles, shall be to one another in a given ratio.

XXII. TH. If a trapezium, inscribed in a circle, have a diameter for one of its sides, the excess of the square of the diameter above the squares of the three remaining sides shall be to the rectangle contained by the first and second of those sides, as the double of the third of them is to the diameter of the circle.

XXIII. PR. To divide the semi-circumference of a given circle into two arches, so that the square of the diameter, together with the square of the chord of one of them, shall be in a given ratio to the square of the chord of the other.

XXIV. Pr. To divide the semi-circumference of a given circle into two arches, so that the square of the diameter shall be in a given ratio to the rectangle contained by the chords of those arches.

XXV. Pr. To divide the semi-circumference of a given circle into two arches, so that the square of the chord of one of them shall be in a given ratio to the rectangle contained by the chord of the other, and the perpendicular drawn to the diameter from the point of division.

XXVI. Pr. To divide the semi-circumference of a given circle into three arches, so that the diameter, and the chords of the three arches, shall be proportionals, according to a given ratio.

XXVII. Pr. In a given semi-circle, to inscribe a trapesium, two of the opposite sides of which shall be equal to each other, and the square of either of them equal to the rectangle contained by the diameter and the side opposite to it.

XXVIII. Pr. To divide the semi-circumference of a given circle into three arches, so that the rectangle, contained by the chords of the first and second of them, shall be equal to the square of the chord of the third.

Book IV.

Prop. I. Pr. The hypotenuse of a right-angled triangle, and the ratio of one of the two remaining sides to the perpendicular drawn to the hypotenuse from the right angle being given, to construct the triangle.

II. Pr. One of the two sides about the right angle of a right-angled triangle being given, and a mean proportional between the other side about the right angle, and the aggregate of that other side and the hypotenuse, to construct the triangle.

III. Pr. The hypotenuse of a right-angled triangle, and the ratio which the aggregate of the hypotenuse, and one of the two remaining sides has to the third side, being given, to construct the triangle.

IV. Th. If the three sides of a right-angled triangle be continual proportionals, and if the greater of the two sides, about the right angle, be cut in extreme and mean ratio, its greater segment shall be equal to the perpendicular drawn to the hypotenuse from the right angle.

V. Th. If the three sides of a right-angled triangle be continual proportionals, the straight line, which bisects the greater of the two acute angles, shall cut the opposite side in extreme and mean ratio.

VI. Pr. To make an isosceles triangle, such that a perpendicular drawn to either of the equal sides from the opposite angle, shall cut that side in extreme and mean ratio.

VII. Pr. From the summit of a given triangle, to draw a straight line to meet the base produced, so that the rectangle contained by that line and one of the two sides shall be in a given ratio to the square of the other side.

VIII. Pr. The angle at the summit of a triangle, the perpendicular drawn from it to the base, and the ratio of the segments into which the perpendicular divides the base being given, to construct the triangle.

IX. Pr. The diameter of a circle described about a triangle, the aggregate of the two sides, and the altitude of the triangle being given, to construct it.

X. Pr. The altitude of a triangle, the rectangle contained by the base, and the aggregate of the two remaining sides, and the diameter of a circle inscribed in the triangle, being given, to construct it.

XI. Pr. The base of a triangle, the perpendicular drawn to it from the opposite angle, and a square equal to the rectangle contained by the two remaining sides, being given, to construct the triangle.

XII. Th. The diameter of a trapezium, which has two parallel sides, divides it into two triangles, which are to one another as these parallel sides.

XIII. Pr. To divide a given trapezium, having two parallel sides, by a straight line passing through a given point of one of them, into two parts, which shall have to one another a given ratio.

XIV. Pr. From a given point, in the base produced, of a given parallelogram, to draw a straight line, meeting the opposite side produced, so as to cut off, from the parallelogram a trapezium, which shall be in a given ratio to the triangle contained by the segment of the opposite side produced, that of the side adjacent to it, and that of the cutting line.

XV. Pr. To inscribe a square in a given triangle.

XVI. Pr. In a given triangle, to inscribe a rectangle, the sides of which shall be to each other in a given ratio.

XVII. Pr. In a given triangle, to inscribe a rectangle, which shall be equal to a given square.

XVIII. Pr. In a given triangle, to inscribe a rectangle, which shall be to the triangle in a given ratio.

XIX. Pr. In a given square, to inscribe an oblong, the two adjacent sides of which shall be to one another in a given ratio.

XX. Pr. In a given square, to inscribe an oblong equal to a given rectilineal figure not greater than the half of the square.

XXI. Pr. In a given square, to inscribe an oblong, which shall be to the square in a given ratio; not greater than that of one to two.

XXII. Pr. In a given square, to inscribe a square which shall be to it in a given ratio, not less than that of one to two.

BOOK V.

I. TH. If, from the right angle of a right-angled triangle, two straight lines be drawn, making equal angles with either of the sides, the straight line, which is made up of the hypotenuse and the exterior segment, is cut harmonically by that side, and the straight line so drawn, which is within the triangle.

II. PR. To find a third harmonic proportional to two given unequal straight lines.

III. PR. To find an harmonic mean proportional between two given unequal straight lines.

IV. PR. To divide a given finite straight line in extreme and mean harmonic ratio.

V. TH. If from the extremities of a given straight line, which is divided harmonically, and from the points of division, straight lines be drawn, all passing through a given point, without the given line, any straight line, which is parallel to one of them, considered as the first, and is terminated by the second and fourth, shall be bisected by the third of them.

VI. TH. If four given straight lines have a point common to them all, and if a straight line, drawn parallel to any one, considered as the first, and terminated by the second and fourth, be bisected by the third of them, any straight line, which cuts all the four given straight lines, shall be divided by them harmonically.

VII. TH. If four given straight lines, having a point common to them all, cut a fifth straight line harmonically, any other straight line, which is cut by the four given straight lines, shall be divided by them harmonically.

VIII. PR. To draw four straight lines, so that any other straight line, which is cut by them all, shall be divided by them harmonically.

IX. Pr. To find four geometrically proportional straight lines, the differences of which shall be harmonic proportionals.

X. Pr. To find a square, which shall be an harmonic mean proportional, between two given unequal squares.

XI. Pr. To find a square, which shall be a third harmonic proportional to two given unequal squares.

XII. Pr. To find four geometrically proportional straight lines, the differences of the squares of which shall be harmonic proportionals.

XIII. Th. If the base of a triangle be produced, and if, from any point in the part produced, a straight line be drawn cutting the one side of the triangle and meeting the other, it shall be cut harmonically by the side of the triangle, and by the straight line drawn from the summit of the triangle, through the intersection of the two straight lines that join the angles at the base, and the points, in which the opposite sides are cut by the line itself so drawn.

XIV. Th. If straight lines be drawn, from the extremities of the base of a triangle, through any the same point of the perpendicular, let fall from the summit on the base, the straight line joining the two points, in which they meet the sides, shall be divided, by the perpendicular, into two parts, that subtend equal angles, at the point where the perpendicular meets the base.

Book VI.

I. Th. If two unequal arches of a circle be each of them less than a quadrant, the greater of them shall have to the less a greater ratio, than the supplement of the less has to the supplement of the greater.

II. Th. If two unequal arches of a circle be each of them less than a quadrant, the tangent of the greater shall have to the tangent of the less, a greater ratio than the greater arch has to the less.

III. TH. If two triangles have their vertical angles equal to one another, and have the two remaining angles in each, neither of which is obtuse, unequal, the side adjacent to the greater angle, in the one triangle, shall have to the base a less ratio, than that which the side adjacent to the corresponding angle, in the other triangle, has to the base.

IV. TH. If, from the greater of two unequal sides of a given triangle, be cut off a part equal to the less, that segment shall have to the remaining segment, a ratio greater than the ratio which the angle adjacent to the remaining segment, has to the angle adjacent to the segment first cut off.

V. TH. If two unequal arches of a circle have a common extremity, the one being a part of the other, and if, from their two other extremities, any two parallels be drawn, to meet the diameter passing through the common extremity, the greater arch shall have to the less, a greater ratio than the parallel terminating the greater arch, has to the parallel terminating the less.

VI. TH. The tangent of the greater of two arches of a circle has to the tangent of the less, a greater ratio than the secant of the greater, has to the secant of the less.

VII. TH. The sine of the greater of two arches of a circle has to the sine of the less, a less ratio than the chord of the greater has to the chord of the less.

VIII. TH. The chord of the greater of two arches of a circle, has to the chord of the less, a less ratio than the versed sine of the greater, has to the versed sine of the less.

IX. TH. If the semi-circumference of a given circle be divided into any two unequal arches, the semi-circumference has to the greater arch, a greater ratio than the less arch has to its sine.

X. TH. If from any point in the diameter, produced, of a semi-circle, two straight lines be drawn to the concave circumference, the greater of them shall have to the less, a less ratio than that, which the greater of the two arches cut off by them, has to the less.

XI. Th. If any given arch of a circle, and its chord, be each of them divided into two unequal parts, by the same straight line drawn from the centre of the circle, the greater segment of the arch shall have to the less a greater ratio, than that which the greater segment of the chord has to the less.

XII. Pr. If the two angles, at the base of a triangle be unequal, the greater has to the less, a greater ratio than that, which the side opposite to the greater has to the side opposite to the less.

XIII. Th. If the two angles at the base of a triangle be unequal, and if a straight line be drawn bisecting the vertical angle, and cutting the base, of the two angles, at the base, the greater has to the less, a greater ratio than that which the greater segment of the base has to the less.

XIV. Th. If a straight line be drawn from the summit of a given isosceles triangle, which divides the angle, at the summit, into two unequal parts, it shall also divide the base into two unequal parts; and the greater segment of the angle, at the summit, shall have to the less a greater ratio, than that which the greater segment of the base has to the less.

XV. Th. If from the summit of a given triangle, a perpendicular be drawn to the base, produced if it be necessary, of the angles which it makes with the other two sides, the greater shall have to the less a less ratio, than that, which the greater segment of the base, or the base produced, has to the less.

XVI. Th. If a perpendicular to the base of a scalene triangle, drawn from its summit, fall within the triangle, of the angles at the base, the greater has to the less, a less ratio, than that which the greater segment of the base has to the less.